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OPTIMAL TRANSFERS BETWEEN CLOSE ELLIPTICAL ORBITS

by J. P. Marec

Office National D'Études et de Recherches Aérospatiales (ONERA)

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When they are quoted in the text, they are only preceded by a chapter number if the chapter is not the one in which they are quoted.

The numbers in brackets refer to the references.

The present publication was the object of a Doctoral Thesis defended 13 June 1967 at the Faculty of Sciences of Paris.

NOTATIONS

A : apogee,
A : accelerating impulse,
A : accelerating thrust,
a : semi-major axis,

$B = \sqrt{\mu/b}$

B : matrix defined in (I,3 - 58),
b : semi-minor axis,
C : characteristic velocity [system (S_1)],

$\Delta\vec{C}$: consummation vector defined in (ii,5 -19),

$\vec{c}$: fixed vector (line vecotr),
D : maneuverability domain,
D : decelerating impulse,
(*D*): directrix curve,
D : decelerating thrust,

$\vec{D}$: thrust direction,
E : energy (per unit of mass),

$\vec{e}$: perigee vector,

$e = |\vec{e}|$: eccentricity,

$\vec{e}_{plan}$: projection of $\vec{e}$ on to the plane of reference, with components α and β $\vec{Ox}$ and $\vec{Oy}$.

$\Delta e_{//}$: composed of $\Delta\vec{e}_{plane}$ parallel to $\Delta\vec{j}$,

$\Delta e_{\perp}$: composed of $\Delta\vec{e}_{plane}$ perpendicular to $\Delta\vec{j}$,

F : thrust,
F : front in the strict sense,
F': front in the broad sense,

$\vec{f}$: second member of the "equations of movement" (column vector),
G : matrix defined in (I,3 - 67),
G : center of mass,

$\vec{g}$: acceleration of gravitation,
H or *H*: Hamiltonian,

$\vec{h}$: kinetic moment (per unit of mass),
I_{sp} : specific impulse,
i : inclination,
J : index of performance [systems (S_2)],

$\Delta\vec{j}$: rotation vector of the plane of the orbit,

K : matrix given in appendix 3,
L : straight ascent (geocentric) or ecliptic longitude (heliocentric),

$L_e = (\vec{Ox}, \vec{p}_e)$
$L_z = (\vec{Ox}, \vec{p}_z)$
(L) : locus of the extremity E of $\vec{e}_{plan}$,
M : average anomaly,
M : mobile,
m : mass,
N : number of revolutions,
n : average movement,
n : number of maximal thrust arcs,
n_1: number of maximal thrust arcs per revolution,
$O(\varepsilon)$: order of $\varepsilon \left(\frac{O|\varepsilon|}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} \text{terminal limit} \neq 0\right)$
$o(\varepsilon)$: order above $\varepsilon \left(\frac{O|\varepsilon|}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} 0\right)$
(O) : osculating orbit,
(O_p) : osculating directrix orbit,
P : power,
P : perigee,
P : pilot mobile,
(P) : efficiency curve,
$\vec{P}$: kinematic adjoint (line vector),
$\vec{p}$: adjoint vector (line vector),
p : "parameter" of the orbit,
$\vec{p}_v$: efficiency vector (line vector),
q : delivery,
$q = p_a/p_e$,
$\vec{R}$: vector defined in (I,3 - 29),
$\vec{R}$: vector radius,
(S): general propulsion system,
(S_1): propulsion system with constant ejection velocity and limited thrust,
(S_2): propulsion system with modulable ejection velocity and limited power,
S : index of performance,
T : period,
(T): trajectory,
ΔT : defined in (II,6 - 5),
-U : potential
$U_{(x)}$: unit step = 1/2 (1 + sign x),
U : "command domain",
$\vec{u}$: command vector,
u : eccentric anomaly, $\left(\mathrm{tg}\,\frac{u}{2} = \sqrt{\frac{1-e}{1+e}}\,\mathrm{tg}\,\frac{v}{2}\right)$.
$\vec{V}$: velocity,

(V): Viviani's Window,
V : point of Viviani's Window,
υ : true anomaly,
W : ejection velocity,
w : $L - L_e$,
$\vec{X}$: "kinematic state" (column vector),
$\vec{x}$: "state" vector (column vector),
$\vec{Z} = \vec{h}/h$,
$\vec{Z}_{plan}$: projection of $\vec{Z}$ on to the plane of reference, with components ξ and η onto $O\vec{x}$ and $O\vec{y}$,
Oxyz : fixed axes,
OXYZ : rotating axes,
$\alpha = \vec{x} \cdot \vec{e}$,
$\beta = \vec{y} \cdot \vec{e}$,
$\vec{\gamma}$: thrust acceleration (column vector),
$\gamma = |\vec{\gamma}|$
Δ : total variation,
δ : difference between the "state" and the "nominal state",
$\delta = (O\vec{x}, \Delta\vec{j})$,
ε : transfer value,
ζ : reduced variation,
$\eta = \vec{y} \cdot \vec{Z}$
Θ : commutation function,
θ : argument,
$\lambda = \Delta X / F_{max} \Delta t$,
Λ : straight ascent (geocentric) or ecliptic (heliocentric) of the perigee,
μ = (constant of universal gravitation) $\times$ (mass of attracting body),
$\xi = \vec{x} \cdot \vec{Z}$
$\vec{\rho} = \vec{r} + \vec{p}_V^{\,t}$,
(Σ): sphere with center M and radius of unity,
τ : state variable replacing time t,
ν : specific variation,
ψ : angle to the center of the thrust arcs,
Ψ : angle of thrust with the local horizontal,
ω : perigee argument,
Ω : straight ascent from the ascending node,
$(\ )_o$: initial,
$(\ )_f$: final,
$\overline{(\ )}$: nominal,
$(\ )^*$: optimal,
$(\ )^t$: transposition of one matrix,
∇ : nabla operator $\frac{\partial}{\partial x_i}$ (line operator).

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OPTIMAL TRANSFERS BETWEEN CLOSE ELLIPTICAL ORBITS

Jean-Pierre Marec

ABSTRACT. The optimal rendezvous and transfers (minimum mass consumption for a given duration) between close elliptical orbits are analyzed in the case of propulsion systems characterized by a constant ejection velocity and a limited thrust; the performance obtained is compared to the performance of propulsion systems characterized by a modulable thrust and a limited power.

The choice of the orbital elements as state components, added to the linearization hypothesis, leads to important simplifications (the adjoint vector is constant, in particular). This gives some general results concerning, among others things, the number of maximum thrust arcs per revolution, the singular cases when the linearized solution is not unique but degenerates into a large number of solutions and the "induction" phenomena (non-imposed variations of some orbital elements, induced by the imposed variations of the other orbital elements).

The solution brings in the notions of "efficiency vector" or "primer vector" (indicating the direction of the optimal thrust), of "directrix orbit" (locus in the absolute axes reference system of the efficiency vector extremity originating at the mobile) and of "efficiency curve" (locus of this extremity in the rotating axes system).

The analytical solution is developed in some particular cases presenting an obvious practical interest:

- optimal infinitely small variation of the semi-major axis,
- optimal infinitely small rotation of the orbital plane,
- optimal transfers between close coplanar circular orbits,
- plane optimal transfers of the Hohmann type between non-intersecting direct, co-axial, close near-circular orbits (higher order analysis),
- optimal transfers and optimal long-duration rendezvous between coplanar or non coplanar, close near-circular orbits.

INTRODUCTION

The problem of optimal transfers and rendezvous between orbits is fundametal in space dynamics. ./5*

These relatively new studies, at one and the same time, call on classical "Celestial Mechanics" [1] and for modern optimization methods [2-6] derived from the "Calculus of Variations".

The analytical results obtained, the very first of which date, back to 1925 [7], concern first of all optimal transfers of *indefinite duration* in a

* Numbers in the margin indicate pagination in the foreign text.

central field of gravitation made by using a propulsion system with a constant ejection velocity, generally capable of furnishing impulses. The initial studies [7-9] assumed the number of impulses fixed in advance, while in recent studies [4,10-19], this number generally constitutes one of the particularly important results [20] of optimization.

The recent progress achieved in this field is essentially due to the use of orbital elements as components of state, of the characteristics of velocity as an independent variable and to the use of the notion of Contensou's "maneuverability domain" [2-4].

The problem of optimal transfers of *fixed duration* between *distant* orbits remains very difficult to resolve, although a certain number of general results have been obtained [21-30].

In the case of propulsion systems with constant ejection velocity, the singular arcs with "intermediate thrust" [31-33] have been the object of particular attention.

On the other hand, in the case of propulsion systems with "weak acceleration" (electrical propulsion), there exist numerous numerical studies [34,35].

However, clear progress in the analytical study of the problem has been made in the relatively favorable case of a propulsion system *with limited power and with modulable ejection velocity*, where Ross and Leitmann [36], and later Edelbaum [37], have given the solution, respectively, of the most general transfer and of the most general rendezvous between *close* elliptical orbits. On the other hand Gobetz has studied the case of transfer between close circular orbits [38] and more recently of rendezvous between close near-circular orbits [39] for the same type of propulsor.

By integration Edelbaum has also been able to extend the results of a linearized study to certain transfers between distant orbits for a large number of revolutions [40].

In the case of a propulsion system with *limited power* and constant ejection velocity (ultimately in the case of a propulsion system capable of delivering impulses), the study remains tricky even for close orbits.

Except for the excellent study of McIntyre and Crocco, concerning transfers between close coplanar circular orbits [41-44], this problem has not been studied very much up to the present time.

The present publication summarizes and completes the results obtained by the author on this subject [45-47]. The case of optimal impulse transfers /6
between close non-coplanar near-circular orbits [47] has been studied in a parallel manner by Gobetz, Washington and Edelbaum [48-49].

The analytical study of infinitesimal transfers is of interest for several reasons.

First of all, such transfers have an intrinsic interest which should not be neglected. They are met very often in practice. They are evidently associated with problems of correcting trajectories [50, 51].

On the other hand, there exists a qualitative and even quantitative agreement which is acceptable between the optimal solution concerning a completed transfer, calculated numerically, and the solution of the linearized problem associated with this transfer, obtained analytically, at least if the departure and arrival orbits are not extremely far apart.

Finally, the linearized solution constitutes an excellent first approximation for the numerical calculation of the optimal solution by successive approximations. Its use considerably reduces the number of iterations and can avoid converging upon uninteresting, local optimums by furnishing on departure a command law very close to the optimal law [55].

* * *

The present study includes two Parts:

The First Part is a general study in which the method of optimization is explained and applied to the problem of optimal infinitesimal transfers.

The Second Part treats a certain number of particular cases for which the analytical study has been more progressive and which offer evident practical interest.

I - GENERAL STUDY

After briefly referring to the method of optimization used, the general problem of optimal transfers is defined and studied particularly in the case of a central gravitational field with slight separations (linearization). A certain number of general results are obtained, notably concerning uncoupling between the optimal variations of the different elements of the orbit. /7

I,1. CONTENSOU'S THEORY OF THE "MANEUVERABILITY DOMAIN" AND PONTRYAGIN'S "MAXIMUM PRINCIPLE"

There is no question of giving here a complete and rigorous demonstration of Pontryagin's "Maximum Principle" as it appears, for example, in [5] and [6], but merely of reviewing its declaration and giving a simplified demonstration of it in the linear case, by referring to Contensou's optimization theory [2-4]. This demonstration is enough for a first order study of transfers between close orbits.

I,1.1. Definitions

Let an evolving system be defined at every instant t by the datum of the "*state*" $\vec{x}$, column vector with n components x_i ($i = 1, 2, \ldots, n$) satisfying the "*equations of movement*":

$$\vec{V} = \dot{\vec{x}} = \vec{F}(\vec{x}, \vec{u}, t) \qquad (1)$$

where $\vec{u}$ is the "*command*" vector with r components u_k ($k = 1, 2, \ldots, r$), arbitrary time functions, not necessarily continuous (which makes the application of the traditional "calculus of variations" tricky), possibly subject to restraints of inequality forcing the vector extremity $\vec{u}$ to belong to a certain "*command domain*" *U*,

$$\vec{u} \in \mathcal{U}. \qquad (2)$$

If the initial state $\vec{x}_o$ is fixed:

$$\vec{x}(t_o) = \vec{x}_o \qquad (3)$$

and if an admissible law of command $\vec{u}(t)$ is given a priori, the integration of the system (1) beginning with the initial instant t_o generally permits a determination of the "trajectory" (*T*), i.e. the state $\vec{x}(t)$ with each later instant t and in particular at the final, fixed instant t_f (the final state $\vec{x}_f$ is supposed to be completely free for the moment). /8

The linear function* of the components of the final state *(index of performance)*:

$$S = \vec{c} \cdot \vec{x}_f \tag{4}$$

(where $\vec{c}$ is a given line vector with n components C_i) then takes on a very definite value.

If now the choice of the law of command $\vec{u}(t)$ in U is arbitrary, S is a function of this law and it is possible to envisage the problem of optimization below:

I,1.2. Statement of the Problem of Optimization

To determine the admissible law(s) of command $\vec{u}(t)$ [i.e. referring to restraint (2)] which *locally* maximize(s) the index of performance S.

For such a law there corresponds an optimal trajectory (T) leading to a point(F_1, F_2 or F_3) of the "frontier in the broad sense" $F'(t_f)$ of the "*accessible domain*" $A(t_f)$ at the moment t_f [2-4] (Figure 1).

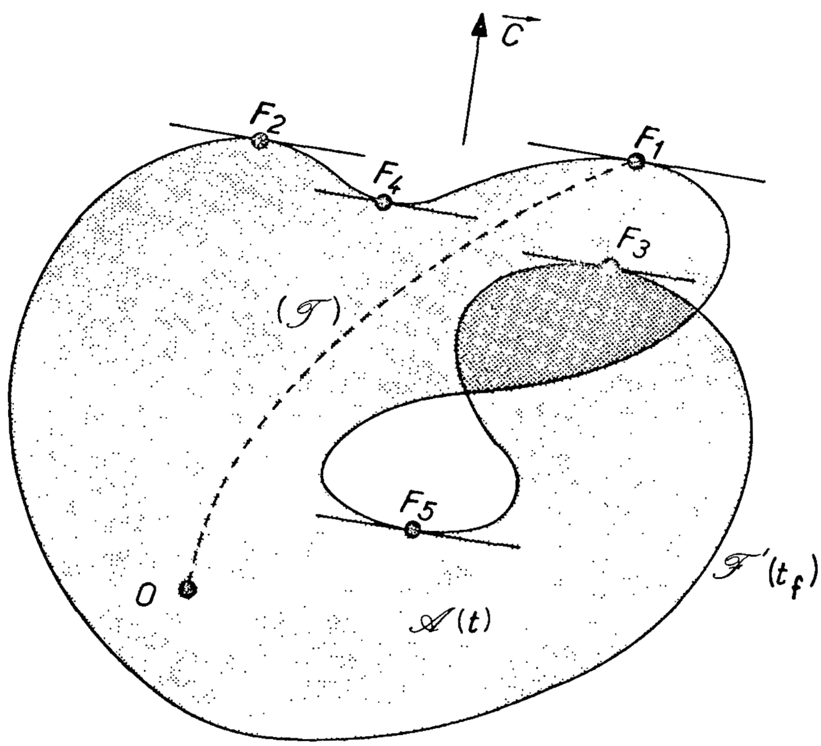


Fig. 1. Accessible Domain.

*This condition is not as restrictive as it appears. It is possible to adapt most problems of optimization to this case.

Then it is necessary to compare directly with each other the different locally optimal solutions such as F_1, F_2 and F_3, but only to retain the one (F_1) which assures the absolute maximum of the index of performance written S. Solutions such as F_3, which do not correspond to a point of the "frontier in the strict sense" $F(t_f)$ are thus automatically eliminated. /9

In general the application of the "maximum principle" introduces supplementary parasitic solutions which correspond to local minima (or to passes) such as F_4 and F_5 [11].

The suitable choice of a trajectory $\overrightarrow{x(t)} + \overrightarrow{\delta x(t)}$ close to the trajectory $\vec{x}(t)$ meeting to one of these points,permits an increase in S to be obtained only *on a higher order* than $|\delta\vec{x}|$.

I,1.3. Simplified Demonstration of the "Maximum Principle".

I,1.3.1. "Maneuverability domain".

The *"maneuverability domain"* $D(\vec{x}, t)$ in the state $\vec{x}$,at the instant t,is made up by the assembly of points $\dot{\vec{x}} = \vec{V}$ of the hodograph space among which the functioning point may be chosen.

The "maneuverability domain" $D(\vec{x}, t)$ is therefore the transform of the "command domain" U defined above, through the transformation of $\vec{u} \Rightarrow \vec{V}$ defined by equation (1).

The "command domain" of Pontryagin's theory is a useful notion for the practical application of the theory of optimization. For the demonstrations and geometrical interpretation of the results,it is preferable to refer to the notion of Contensou's "maneuverability domain".

The choice of $\vec{V}$ inside the "maneuverability domain" is completely free. Therefore it is particularly independent of previous choices: *the vector* $\vec{V}$ *is not limited to any kind of continuity.*

On the other hand, we presuppose,as fulfilled,the conditions of regularity required in order to write the different equations which follow.

I,1.3.2. Statement of the "Maximum Principle".

A *necessary* condition for the trajectory $\vec{x}$ (t) to be optimal is that at every moment the vector $\vec{V}$ is chosen in the domain $D(\vec{x}, t)$ in such a way that the Hamiltonian:

$$H = \vec{p} \cdot \vec{V} \tag{5}$$

is maximal. $\vec{p}$ is a line vector with n components p_i, "*adjoint*" to the state $\vec{x}$, defined in the following way:

$\vec{p}$, $\vec{x}$ and t being given, the maximal value $H^* = \vec{p} \cdot \vec{V}^*$ of H in reference to $\vec{V}$ is only a function of $\vec{p}$, $\vec{x}$, t (Figure 2).

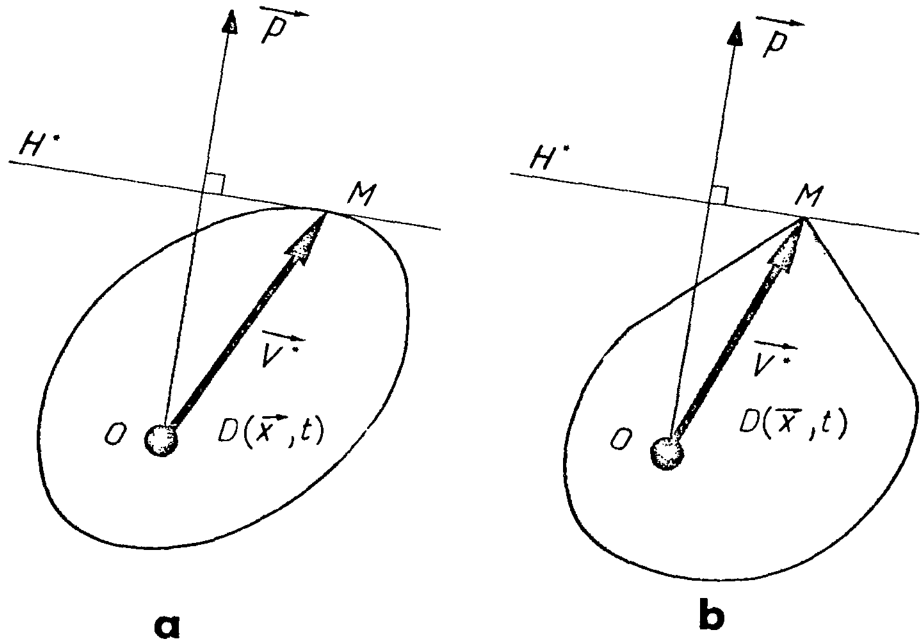


Fig. 2. Maneuverability Domain. Maximum Condition. /10

The adjoint vector $\vec{p}$ must satisfy the "*adjoint system*":

$$\dot{\vec{p}} = -\nabla . H^* \quad \left(\nabla = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right] \right) \tag{6}$$

and the final conditions:

$$\vec{p}_f = \vec{c} \tag{7}$$

The maximum condition shows that only some points of the front of the maneuverability domain D are used, and more precisely points common to the frontier of the D domain and the small convex contour encircling D (Figures /11

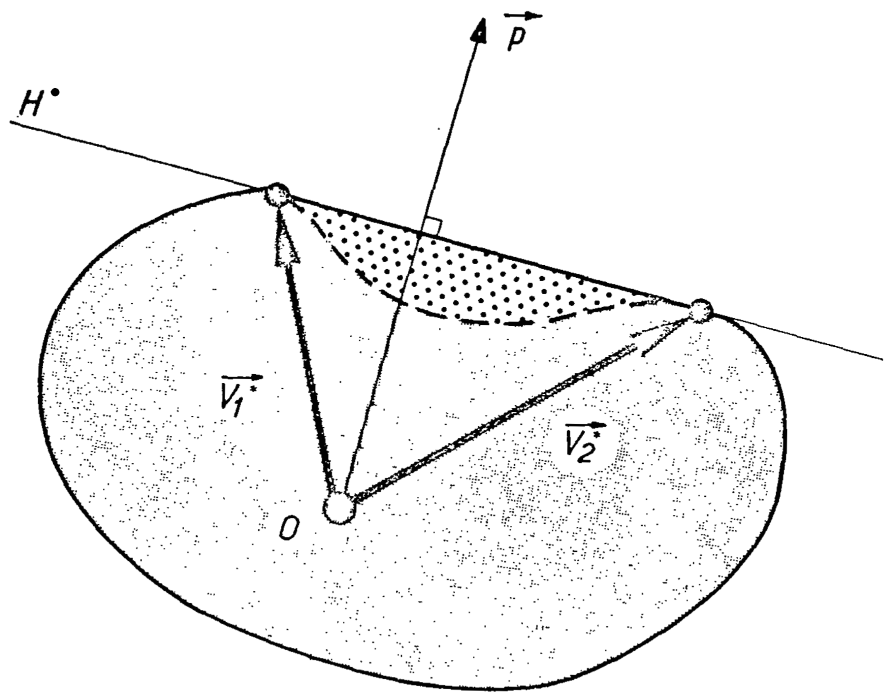


Fig. 3. "Convexization" of the Maneuverability Domain.

I,1.3.3. Simplified demonstration.

Let there be a trajectory $\vec{x}(t)$, *not necessarily optimal*, corresponding to the choice $\vec{V}(t)$ of $\vec{V}$ inside the domain $D(\vec{x}, t)$ (Figure 4). It is possible to define an adjoint $\vec{p}(t)$, as in the preceding paragraph, by integrating backward equation (6) starting from the conditions of (7); (H^* is replaced by H and $\vec{V}^*$ by $\vec{V} + \delta^{\to}V_1$).

Let $\vec{x}(t) + \delta\vec{x}(t)$ be a neighboring trajectory. We propose the evaluation:

$$\frac{d}{dt}(\vec{p}\cdot\delta\vec{x}) = \dot{\vec{p}}\cdot\delta\vec{x} + \vec{p}\cdot\delta\vec{V} = -(\nabla.H)\cdot\delta\vec{x} + \vec{p}\cdot\delta\vec{V_1} + \vec{p}\cdot\delta\vec{V_2} + \vec{p}\cdot\delta\vec{V_3} \tag{8}$$

$\delta^{\to}V_2$ can be great (commutation, Figure 6), but

$$\vec{p}\cdot\delta\vec{V_2} = \delta H = (\nabla.H)\,\delta\vec{x} + o\left(|\delta\vec{x}|\right) \tag{9}$$

where $o(\varepsilon)$ signifies: order greater than ε $\left(\frac{o(\varepsilon)}{\varepsilon} \xrightarrow[\varepsilon\to 0]{} 0\right)$ Therefore:

$$\frac{d}{dt}(\vec{p}\cdot\delta\vec{x}) = \vec{p}\cdot\delta\vec{V_1} + \vec{p}\cdot\delta\vec{V_3} + o\left(|\delta\vec{x}|\right) \tag{10}$$

where $\vec{p}\cdot\delta\vec{V_1} \geq 0$ and $\vec{p}\cdot\delta\vec{V_3} \leq 0$ by definition (Figure 4).

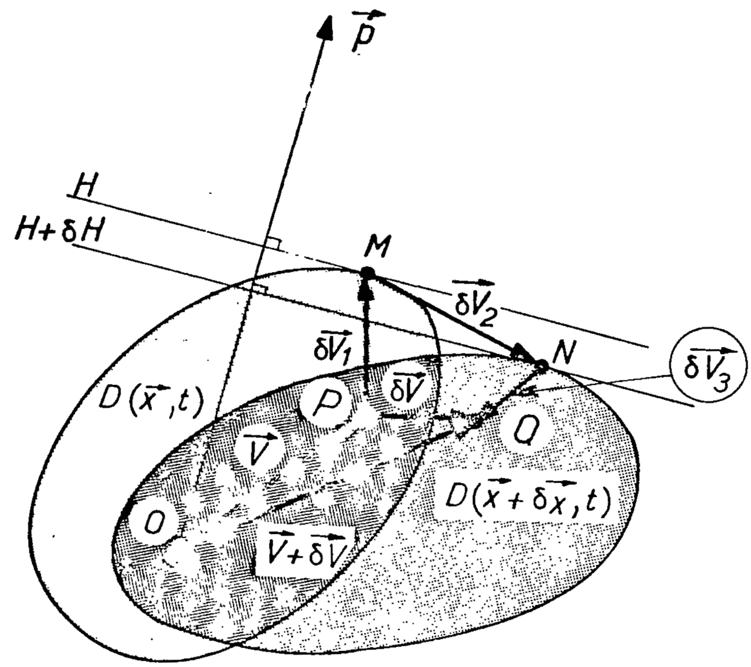


Fig. 4. Maneuverability Domains Relating to Two Neighboring States.

The maximum condition is a *necessary condition* for trajectory $\vec{x}(t)$ to be optimal. /12

Let us just call t_2 the last moment beyond which the maximum condition ($\delta\vec{V}_1 = 0$) is always realized $(t_o = 0 < t_2 \leq t_f = 1)$ and t_1 a moment slightly before ($t_2 - t_1 = \tau$ small) (Figure 5).

Beginning with the instant t_1 let us choose a neighboring trajectory such that $\delta\vec{V}_3 = 0$ and let us integrate (10) from t_1 to t_f. The increase in the index of performance S is:

$$\delta S = \vec{c}\cdot\delta\vec{x}_f = \vec{p}_f\cdot\delta\vec{x}_f = \underbrace{\vec{p}_1\cdot\delta\vec{x}_1}_{=0} + \int_{t_1}^{t_2}\vec{p}\cdot\delta\vec{V}_1 dt + \int_{t_1}^{t_f} o\left(|\delta\vec{x}|\right) dt = \tag{11}$$

$$\tau\left(\vec{p}\cdot\delta\vec{V}_1\right)_{\text{mean for } t_1,t_2} + \int_{t_1}^{t_f} o\left(|\delta\vec{x}|\right) dt .$$

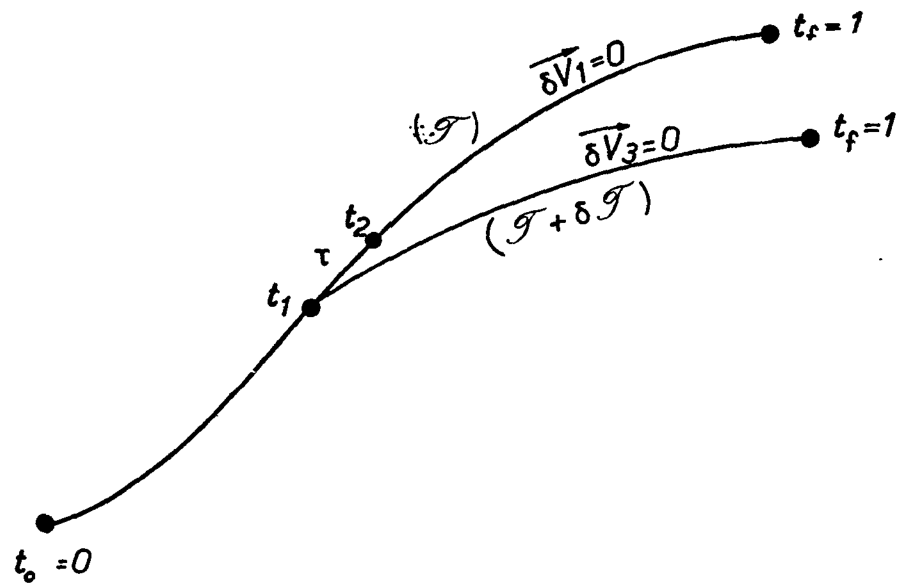


Fig. 5. Demonstration of the Necessary Condition.

Since the evaluation of this last integral is tricky in a general case, the necessary condition will only be demonstrated in the simple cases of the "*linear systems*" of the type:

$$\dot{\vec{x}} = \vec{F} = a(t)\,\vec{x} + \vec{b}(\vec{u}, t) \tag{12}$$

where a(t) is a tensor.

In this case the integral is zero, for H is linear in $\vec{x}$ when we pass from point M to point N of Figure 4. As a matter of fact $\vec{u}$ is the same in both of these points, therefore $\vec{F}$ is linear in $\vec{x}$.

Thus

$$\delta S = \tau\left(\vec{p}\cdot\overrightarrow{\delta V_1}\right)_{\text{by mean for } t_1, t_2} > 0 \tag{13}$$

Therefore S can be increased by choosing a neighboring trajectory. Therefore the initial trajectory is not optimal. /13

In general the maximum condition is not a *sufficient condition.*

Let us just suppose that it is fulfilled at every instant, i.e. that $\delta\vec{V}_1 \equiv 0$.

The integration of (10) from t_o to t_f gives:

$$\delta S = \vec{p_f} \cdot \delta\vec{x_f} = \underbrace{\vec{p_o} \cdot \delta\vec{x_o}}_{=0} + \int_{t_o}^{t_f} \vec{p} \cdot \delta\vec{V_3}\, dt + \int_{t_o}^{t_f} o\left(|\delta\vec{x}|\right) dt. \tag{14}$$

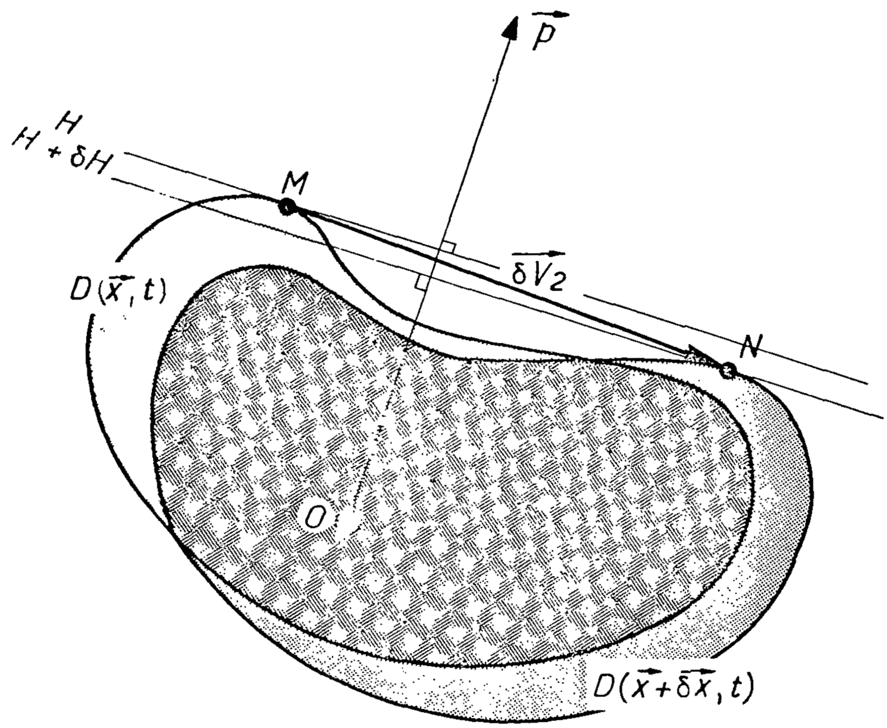


Fig. 6. Commutation.

We shall consider a neighboring trajectory obtained by choosing $\delta\vec{V}_3$ so that $|\vec{p} \cdot \delta\vec{V_3}| = o\left(|\delta\vec{x}|\right)$. This is possible, for example, by taking the point Q of Figure 4 on the surface of the domain $D(\vec{x}+\delta\vec{x}, t)$ and near point N, on the portions of the trajectory corresponding to the case of Figure 2a, and by taking Q and N on the portions corresponding to the case of Figure 2b. The two integrals of (14) can be of the same order. We see that S can undergo a variation δS of which the sign is not determined and can particularly be increased $\delta S > 0$, but only *on an order greater than* $|\delta\vec{x}|$. In general the application of the maximum principle can lead to parasitic solutions such as F_4 and F_5 (Figure 1).

However, the maximum condition is a sufficient condition (therefore a *necessary and sufficient* condition) when the second integral of (14) is zero. /14

Thus choosing $\delta\vec{V}_3$ in any admissible manner and so that the neighboring trajectory $\vec{x}(t) + \delta\vec{x}(t)$ is not too far away from the nominal trajectory $\vec{x}(t)$, equation (14) shows that δS is always < 0 since $\vec{p} \cdot \delta\vec{V}_3$ is always ≤ 0.

Therefore the maximum condition is necessary and sufficient for the "*linear systems*" of type (12).

I,1.3.4. Conclusion.

A problem of optimization stated in §I,1.2. is thus resolved in this way: the application of the maximum condition (absolute maximum H at every instant with reference to $\vec{V} \in D$ *therefore in reference to* $\vec{u} \in U$) furnishes the optimal command $\vec{V}^*$ or $\vec{u}^*$ which is carried into the system (1) of the equations of movement and into the adjoint system (6). These 2n differential equations are integrated, consideration being given to the 2n conditions with limits (3) and (7).

It is important to note that if the component x_{if} of the final state is *indifferent* (therefore is not fixed and does not take a part in S), the component C_i corresponding to $\vec{C}$, and therefore the component p_{if} corresponding to the final adjoint $\vec{p}_f$, is zero.

On the other hand it is possible to demonstrate that if the component x_{if} of the final state is *fixed*, the component p_{if} corresponding to the final adjoint is not imposed *a priori*. The condition i of (7) is replaced by the condition: x_i fixed, which does not modify the number of conditions at the limits.

I,1.4. Interpretation of the Adjoint Vector

Equations (11) and (14) show that at every moment the adjoint is the *coefficient of influence* of a variation $\delta\vec{x}$ of the state on the index of performance S. If certain components written x_{if} of the final state are fixed, it is true only if the variation $\delta\vec{x}$ is *compatible with these final restraints.*

Let us note that the adjoint vector $\vec{p}$ is only defined by an approximate multiplicative factor by the homogeneous system (6) and the maximum S condition. As a matter of fact it is equivalent to maximizing S or kS(k > 0 arbitrary), i.e. to choosing $\vec{c}$ or $k\vec{c}$. Therefore equation (7) constitutes the normalization condition of the adjoint $\vec{p}$.

I,1.5. First Integral of the Hamiltonian.

In §I,1.3.2. we saw that the optimal Hamiltonian H* is a function of $\vec{p}$, $\vec{x}$ and t. We propose to calculate its derivative with respect to time in an optimal trajectory:

$$\frac{dH^*(\vec{p},\vec{x},t)}{dt} = \frac{d\vec{p}}{dt}\cdot\vec{V^*} + (\nabla . H^*)\cdot\frac{d\vec{x}}{dt} + \frac{\partial H^*}{\partial t} \tag{15}$$

from which, taking (6) into consideration: /15

$$\frac{dH^*}{dt} = \frac{\partial H^*}{\partial t}. \tag{16}$$

When the maneuverability domain D($\vec{x}$, t) does not depend on time, $\frac{\partial H^*}{\partial t} = 0$ we get the Hamiltonian first integral:

$$H^* = C^{te}. \tag{17}$$

* * *

The method of optimization described above is going to be applied at present to the problem of optimal transfers and more exactly to the problem of close optimal transfers (linear case).

I,2. PROBLEM OF OPTIMAL TRANSFERS

Before approaching the particular case of transfers between close orbits, it is useful to provide a few results about the general problem of optimal transfers.

After defining the notion of transfer and analyzing the different propulsion systems, Maximum Principle is applied to optimal transfers in any field of gravitation, which clearly illustrates the differences between the solutions corresponding to varied propulsion systems.

However,all of these solutions call on the same notion of "vector efficiency" $\vec{p}_V$ (Lawden's "first vector" [21]), indicating the optimal direction of thrust, which is widely used and interpreted geometrically: this vector has its origin in the mobile *M* and as its extremity a "pilot mobile *P*, close to *M*, subject to the same thrust acceleration and to the same gravitational field as *M*, and describing a "directrix curve" (*D*) close to the trajectory (*T*) of *M*.

When the gravitational field does not depend on time, the case of transfers without rendezvous is envisaged.

The hypothesis of the central field is then made to allow further advance in the analytical study, In this case the benefit of employing elements of the Keplerian orbit (O) osculating to the trajectory at every instant as components of state is evident. The notion of "directrix orbit" (O_p) gives an interesting interpretation of the law of optimal thrust.

I,2.1. Definitions.

In a very general way we call "*transfer*" (with rendezvous of a mobile /16 M of variable mass subject to a gravitation field $\vec{g}(\vec{r},t) = \overrightarrow{grad}\, U(\vec{r},t)$ given as derived from the potential $-U(\vec{r}, t)$ and to a thrust acceleration $\vec{\gamma}$ (command) any change of position $\vec{r}$ and of velocity $\vec{V}$ of this mobile between two fixed moments t_o and t_f (Figure 1), the change produced by the simple natural movement ($\vec{\gamma} = 0$) capable of being qualified as a zero cost transfer.

If the change $\vec{g}$ does not depend on the time t, it is possible to define generalized "orbits" corresponding to the natural movements of mobiles M launched from initial positions $\vec{r}_o$ with initial velocities $\vec{V}_o$, such an "orbit" depending only on the initial conditions $\vec{r}_o$, $\vec{V}_o$. It is then possible to define transfers *without rendezvous*, when only the "final orbit" at the instant t_f is imposed, without the final position M_f of the mobile on this "orbit" being imposed.

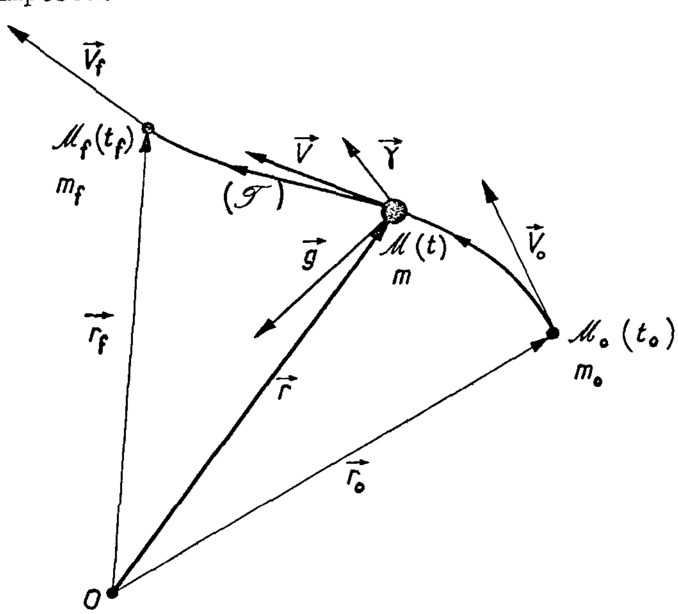


Fig. 1. Transfer.

The state $\vec{x}$ of the mobile M at the instant t can be defined by the data of the kinematic elements $\vec{r}$, $\vec{V}$ and of mass m. Therefore transfer is the passage of the state $\vec{r}_o$, $\vec{V}_o$, m_o at instant t_o to the state $\vec{r}_f$, $\vec{V}_\varepsilon$, $\vec{m}_f$ at the instant t_f (the final "kinematic" state $\vec{r}_f$, $\vec{V}_f$ possibly being partially indetermined).

It is obviously desirable to achieve the transfer previously defined in the most economical manner possible, economy being capable of definition as a function of a larger or small number of criteria.

The criterion most often retained is that of the economy of the propulsive ejected by the rocket to produce the thrust acceleration $\vec{\gamma}$. As a matter of fact let us return to the propulsion equation:

$$F = qW \tag{1}$$

where F is the thrust force, $q = -\dot{m}$, the feed of the propulsive, and W the velocity of ejection.

Thus it is a matter in each mission envisaged of minimizing the consumption of propulsive $|\Delta m| = |m_f - m_o|$ for a fixed time period $\Delta t = t_f - t_o$.

I,2.2. Propulsion Systems

Without going into detail on present and future propulsion systems, we here have a matter of defining a simple mathematical model, unique if possible, describing all of them and capable of being adapted to each particular case by a simple modification of parameters. /17

In the proposed model (S), the choice of the direction $\vec{D}$ of thrust is supposed to be entirely free and the two parameters of command remaining F (magnitude of thrust) and q (feed) must be chosen inside the "command domain" *U* represented in Figure 2a.

In this model (S) the jet power:

$$P = \frac{1}{2}(-\dot{m})\, W^2 = \frac{FW}{2} = \frac{F^2}{2q} \tag{2}$$

is limited:

$$P \leqslant P_{max} \tag{3}$$

whence the arc of frontier parabola $\widehat{AB}$ (or $\widehat{AC}$).

The ejection velocity:

$$W = \frac{F}{q} \tag{4}$$

as well as the feed q have upper limits:

$$W \leqslant W_{max} \tag{5}$$

$$q \leqslant q_{max} \tag{6}$$

whence the rectilinear fronts OA and BD.

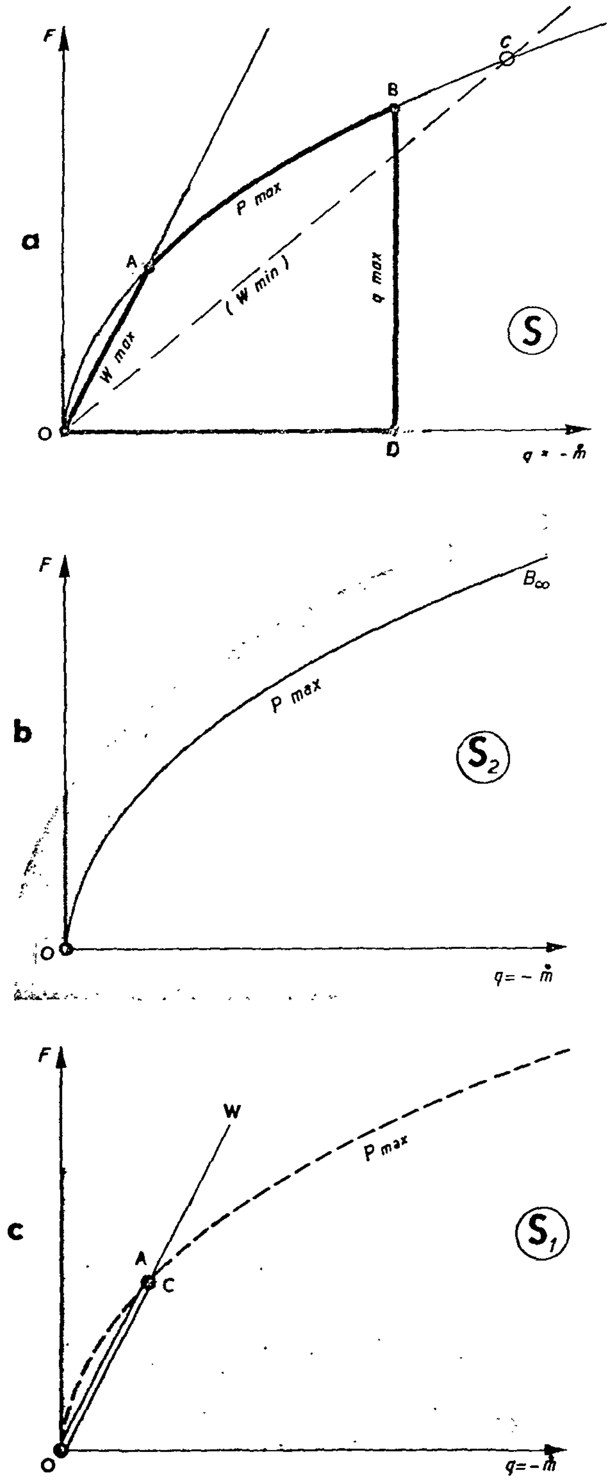


Fig. 2. Propulsion Systems.

It is likewise possible to impose a lower limit $W_{min} > 0$ to the ejection velocity

$$W \geqslant W_{min} \qquad (7)$$

which adds a rectilinear frontier OC. Point C can be found on the arc $\widehat{AB}$ or outside of this arc; the theoretical study is the same.

Let us note that the magnitude F of thrust is automatically limited at the top:

$$F \leqslant F_{max} = min\left(F_B, F_C\right). \qquad (8)$$

This model (S) is well adapted to electrical propulsion systems evolved where the jet power is limited by the power of the electric generator, where the ejection velocity, modulable to a certain degree, cannot exceed a certain maximal value (nor often go below a certain minimal value) *for a* /18
definite ejector under penalty of seeing the efficiency of the ejector considerably diminished, and finally where the propulsive feed is limited by the pump system (or even by the ion production system in the case of an ionic propulsor).

The model (S_2) which is derived from model (S) by making

$$W_{max} = \infty \ , \ q_{max} = \infty \ , \ W_{min} = 0 \quad (9)$$

correspond to an *idealized* electric propulsor (Figure 2b), often considered in theoretical studies, where the only limitation bears on the jet power.

Finally the model (S_1) which is derived from the model (S) by making:

$$W_{max} = W_{min} = W \tag{10}$$

corresponds at one and the same time to electric propulsors with a non-modulable ejection velocity and to traditional chemical propulsors (liquids, powders) or to a nuclear rocket (Figure 2c). Thus these different types of propulsors are distinguished by orders of magnitude of specific impulse $I_{sp} = W/g$ and of maximal thrust F_{max}.

While for traditional chemical propulsors specific impulse is of the order of 300 s (powders) or 400 s (liquids) and for the nuclear rocket of the order of 800 s, for electric propulsors specific impulse can reach very high values (10,000 s and even more).

On the other hand, maximal thrust acceleration $\gamma_{max} = F_{max}/m$, which is of g or several g for traditional propulsors, (systems called: "*high thrust*") is very weak (10^{-4} to 10^{-3} g) for electric propulsors (systems called: "*low thrust*").

I,2.3. Case of any Field of Gravitation.

The state ($\vec{r}$, $\vec{V}$, m) and the command ($\vec{D}$, F, q) having been defined in the above paragraphs, let us apply the Maximum Principle to the problem of transfer envisaged above, consisting of passage from the state $\vec{r}_o$, $\vec{V}_o$, m_o at a fixed instant t_o to the state $\vec{r}_f$, $\vec{V}_f$, m_f at a definite instant t_f ($\vec{r}_f$, $\vec{V}_f$ may be partially undetermined) with the minimum consumption $|\Delta m| = |m_f - m_o|$.

The "equations of movement" are very simple:

(properly named equations of movement)

$$\dot{\vec{r}} = \vec{V} \tag{11}$$

$$\dot{\vec{V}} = \vec{\gamma} + \vec{g}(\vec{r}, t) = \frac{F}{m}\vec{D} + \vec{g}(\vec{r}, t) \tag{12}$$

(consumption equation)

$$\dot{m} = -q \tag{13}$$

Then the Hamiltonian is written:

$$H = \vec{p_r} \cdot \dot{\vec{r}} + \vec{p_V} \cdot \dot{\vec{V}} + p_m \dot{m} = \vec{p_r} \cdot \vec{V} + \vec{p_V} \cdot \left(\frac{F}{m}\vec{D} + \vec{g}\right) - p_m q \tag{14}$$

where $\vec{p}_r$, $\vec{p}_V$, p_m are adjoint to elements $\vec{r}$, $\vec{V}$, m of state.

The performance index to be maximized is the final mass: /19

$$S = m_f . \tag{15}$$

Therefore it is necessary to maximize the Hamiltonian at every instant in reference to the command $\vec{D}$, F,q.

The optimal direction $\vec{D}^*$ of the thrust is obviously that of the efficiency vector $\vec{p}_V$:

$$\vec{D}^* = \vec{p_V}/|\vec{p_V}| . \tag{16}$$

Then the Hamiltonian becomes:

$$H = -p_m q + \frac{|\vec{p_V}|}{m} F + \text{(terms independent of q and F)}. \tag{17}$$

The adjoint system in which the optimal direction of thrust has been carried is written:

$$\dot{\vec{p_r}} = -\vec{p_V} \cdot \overline{\overline{\left[\nabla^T . \vec{g}^T\right]}} \qquad \left(\nabla = \frac{\partial}{\partial x_i}, \text{ line operator}\right) \tag{18}$$

$$\dot{\vec{p_V}} = -\vec{p_r} \tag{19}$$

$$\dot{p}_m = \frac{|\vec{p_V}| F}{m^2} . \tag{20}$$

From this is deducted:

$$\frac{d}{dt}(m p_m) = \dot{m} p_m + m \dot{p}_m = \frac{F}{m}\left(|\vec{p_V}| - \frac{m p_m}{W}\right). \tag{21}$$

In succession we envisage the cases of the propulsion systems (S), (S_2) and (S_1), thus completing the results obtained in [22].

1,2.3.1. Propulsion system (S).

At any instant t, H = C^{te} is the equation of a *straight line* of the plane q, F, of fixed slope $mp_m/|\vec{p}_v|$ and with ordinate originally proportional to H (the coefficient of proportionality $m/|\vec{p}_v|$ being positive). Therefore maximizing H comes down to maximizing this ordinate at its origin by choosing the point of functioning q, F suitably. The result depends on the slope of the straight line (Figure 3):

1. If $\frac{mp_m}{|\vec{p_v}|} > W_{max}$, point 0 should be used (zero thrust, ballistic arc).

From (21) we deduce that:

$$mp_m = C^{te} \ [27]. \tag{22}$$

Thus slope $mp_m/|\vec{p}_v|$ varies only by $|\vec{p}_v|$.

2. If $\frac{mp_m}{|\vec{p_v}|} = W_{max}$, it is indeterminate.

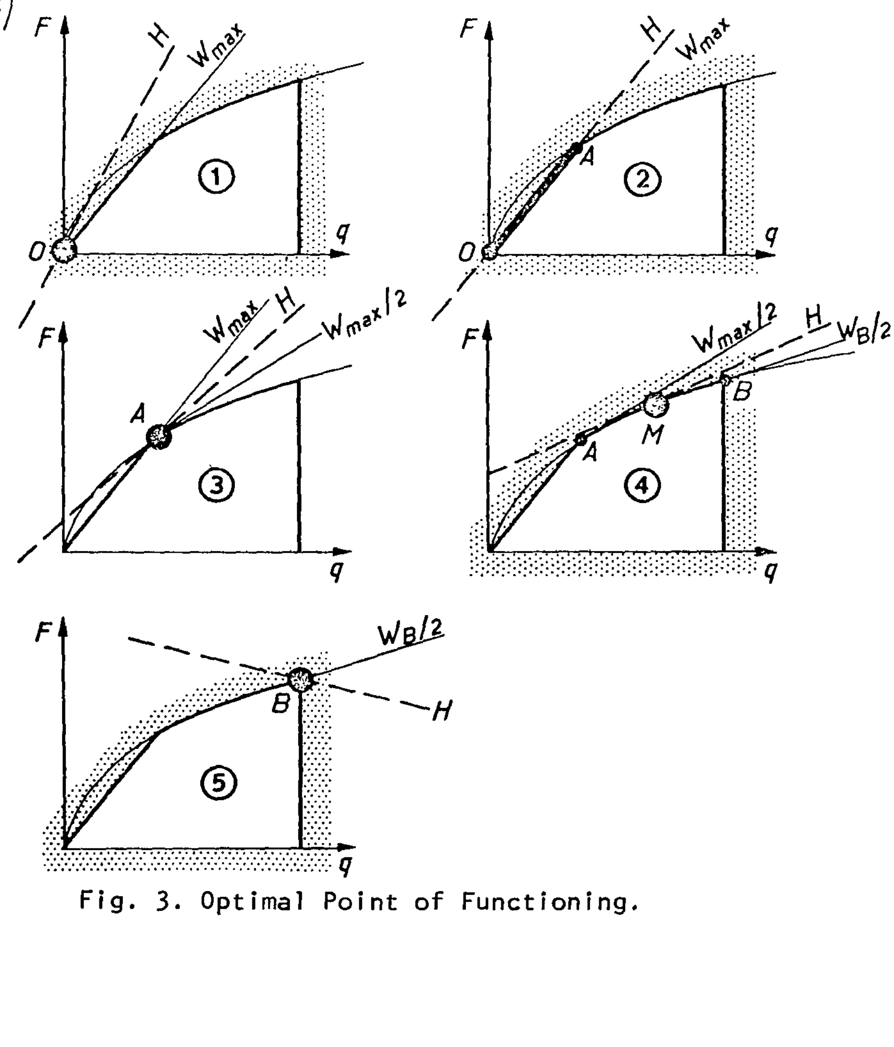


Fig. 3. Optimal Point of Functioning.

This indetermination is not annoying if the equality only occurs at instant t (commutation). On the other hand if the equality takes place during a finite interval of time $t_1 t_2$ the amplification of the Maximum Principle does not furnish the command since every point of segment OA is a priori suitable. Command is obtained directly by writing that the slope is constant for $t_1 t_2$ and equal to W_{max} (singular solution [31-33]) or: /20

$$|\vec{p_V}| \equiv \frac{m p_m}{W_{max}} . \tag{23}$$

From (21) the first integral is therefore deduced:

$$m p_m = C^{te} [27] \tag{24}$$

from which, by (23),

$$|\vec{p_V}| = C^{te} [27] \tag{25}$$

and finally the first integral:

$$\vec{p_V} \cdot \dot{\vec{p_V}} = - \vec{p_V} \cdot \vec{p_r} = 0 . \tag{26}$$

Therefore the vectors $\vec{p}_V$ and $\vec{p}_r$ are orthogonal.

The corresponding arcs are called: "intermediate thrust arcs".

3. If $\frac{W_{max}}{2} \leqslant \frac{m p_m}{|\vec{p_V}|} < W_{max}$ point A must be used (constant thrust F_A). /21

As $\frac{d}{dt}(m p_m) > 0, m p_m$ increases. [sic]

4. If $max\left(\frac{W_B}{2}, \frac{W_C}{2}\right) \leqslant \frac{m p_m}{|\vec{p_V}|} \leqslant \frac{W_{max}}{2}$ point M of the parabola arc $\widehat{AB}$ (or $\widehat{AC}$) must be used (modulated thrust $F = \frac{P_{max}}{m p_m} \left|\vec{p_V}\right|$). Then the first integral is deduced from this:

$$m^2 p_m = C^{te} [27] \tag{27}$$

then $\vec{\gamma} = \frac{\vec{F}}{m}$ *is proportional to the vector of efficiency* $\vec{p}_V$.

5. Finally if $\frac{m p_m}{|\vec{p_V}|} < max \left(\frac{W_B}{2}, \frac{W_C}{2}\right)$ point B (or C) must be used (constant thrust F_{max}).

Thus an optimal trajectory, except for singular cases, is composed of a succession (when the slope $mp_m/|\vec{p}_V|$ varies) of ballistic arcs (F = 0), of constant thrust arcs at maximal ejection velocity ($F = F_A$), of modulated thrust arcs $[F_A \leqslant F \leqslant min(F_B, F_C)]$ and of maximal thrust arcs $[F = min(F_B, F_C)]$.

I,2.3.2. Particular case of propulsion systems (S_2).

In this case it is convenient to define a position of M on the $\widehat{OB}\,\infty$ arc (Figure 2b) by the datum of the thrust F or rather of the thrust acceleration $\gamma = F/_m$.

This choice is of particular interest; as a matter of fact the integration of the equation:

$$-\frac{\dot{m}}{m^2} = \frac{1}{2P_{max}}\gamma^2 \tag{28}$$

between the times t_o and t furnishes:

$$\frac{1}{m} - \frac{1}{m_o} = \frac{J}{P_{max}} \tag{29}$$

where

$$J = \frac{1}{2}\int_{t_o}^{t}\gamma^2 dt \tag{30}$$

is a performance index which increases monotonously when the mass m diminishes.

Therefore this index can advantageously replace the mass m in order to express the expense of the operation.

The new equation of consumption is:

$$\dot{J} = \frac{\gamma^2}{2}. \tag{31}$$

The equations of movement (11), (12) and (31) only contain the command $\vec{\gamma}$.

The canonical transformation: $(m; p_m) \Longrightarrow (J; p_J)$ is such that [25,30]:

$$p_m\, dm = p_J\, dJ \left(= -p_J \frac{P_{max}}{m^2} dm\right) \tag{32}$$

whence:

$$p_m = -p_J \frac{P_{max}}{m^2}. \qquad (33)/22$$

Now, since J does not occur in the second members of the equations of movement, $\dot{p}_J = 0$ and

$$p_J = C^{te} = p_{J_f} = c_J = -1 \qquad (34)$$

since this is a matter of maximizing $S = -J_f$.

Therefore we again find the first integral (27) where *by choice* (34), the constant is equal to P_{max} and the optimal acceleration is given simply by:

$$\boxed{\vec{\gamma} = \vec{p_V}} \qquad (35)$$

The thrust acceleration is equal to the vector of efficiency.

1,2.3.3. Particular case of propulsion systems (S_1).

Since the ejection velocity W is constant, the integration:

$$-W \frac{\dot{m}}{m} = \frac{F}{m} = \gamma \qquad (36)$$

between the moments t_o and t furnishes:

$$-W \, Log \frac{m}{m_o} = C \qquad (37)$$

where

$$C = \int_{t_o}^{t} \gamma \, dt \qquad (38)$$

is the "*characteristic velocity*" which increases monotonously when the mass m diminishes and which can therefore advantageously replace the mass m to express the cost of the operation.

The canonical transformation $(m; p_m) \Longrightarrow (C; p_c)$ is such that:

$$p_m \, dm = p_c \, dC \left(= -p_c W \frac{dm}{m}\right) \qquad (39)$$

from which

$$p_m = -\frac{W}{m} p_c . \qquad (40)$$

The thrust law is therefore given by

$$F = F_{max}\, U(\Theta) \qquad (41)$$

where

$$U(\Theta) = \frac{1 + \text{signe}\,\Theta}{2}$$

is the degree of unity and:

$$\Theta = |\vec{p_v}| + p_c \qquad (42)$$

is the *commutation function.*

The thrust law is one of "all or nothing" ($F = \begin{cases} F_{max} \\ 0 \end{cases}$ according to which $\Theta \gtrless 0$).

On the ballistic arcs, $mp_m = C^{te}$ [equation (22)], and therefore $p_c = C^{te}$. On the propulsed arcs $\frac{d}{dt}(mp_m) > 0$, therefore mp_m increases and p_c decreases.

Optimal Ejection Velocity. /23

The optimal realization of a mission given with the aid of a propulsion system (S_2) leads to a utilization of thrusts such that the representative point M_2 of the command diagram describes, for example, arc $\overset{\frown}{QR}$ (Figure 4).

Realization of the *same mission* with a propulsion system (S_1) of the *same installed power* P_{max} and of constant ejection velocity W lead to a utilization of the points O or M_1.

This constant ejection velocity W (generally) remains at the choice of the user (choice of the ejector to adapt to the energy conversion system).

The problem is choosing the velocity W in the optimal manner (W*), that is to say such that the consumption of propulsive is minimal for the given mission [35]. The corresponding optimal propulsion system (S_1) will be noted (S_1^*).

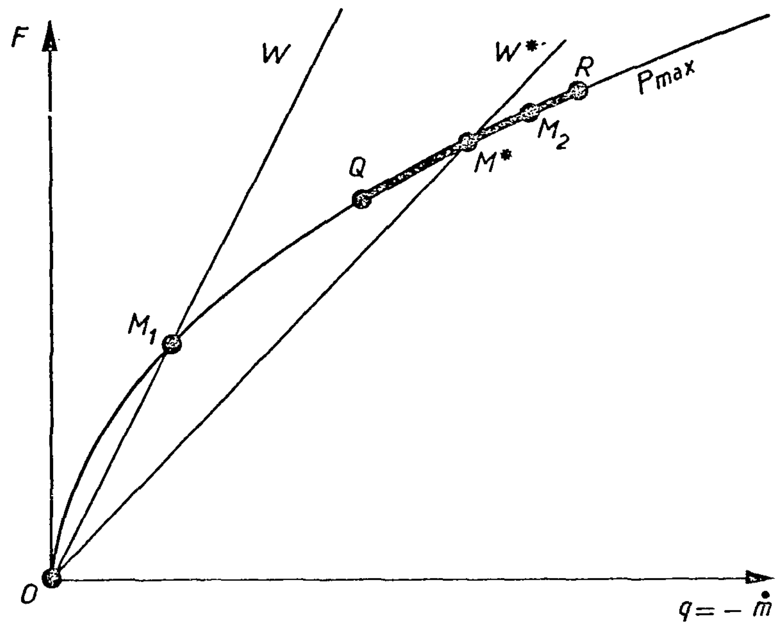


Fig. 4. Optimal Ejection Velocity

It is demonstrated in Appendix 1 that the optimal ejection velocity W* is furnished by the equation:

$$\int_{t_o}^{t_f} \frac{U(\Theta)}{m} \left(\Theta + p_c\right) dt = \int_{t_o}^{t_f} \frac{U(\Theta)}{m} \left(|\vec{p_V}| + 2p_c\right) dt = 0 . \tag{43}$$

Condition (43) is interpreted simply (Figure 5):

Only the part of the efficiency $|\vec{p}_V|/m$ above the commutation level $-p_c/m$ intervenes, which is logical. For the optimal ejection velocity W*, the area (1) representing to some extent the global excess of the efficiency above the commutation level is equal to area (2) below the commutation level.

Examples of the determination of the optimal ejection velocity W* will be given in §II, 1.3.1.3. and II,2.3.2.3.

I,2.3.4. Kinematic interpretation of the vector of efficiency. /24

We have seen the fundamental role played by the vector of efficiency $\vec{p}_V$ in the determination of the optimal thrust law, no matter what system of propulsion is envisaged.

We propose finding a kinematic interpretation of this vector. For it let us find the locus of the extremity P of the vector $\vec{MP} = \vec{p}_V^{\,t}$ in the fixed axes where the movement of the mobile is located (Figure 6). Let us posit $\vec{OP} = \vec{\rho}$. It becomes:

$$\ddot{\vec{\rho}} = \ddot{\vec{r}} + \ddot{\vec{p}}_V^{\,T} \tag{44}$$

now, according to equations (18) and (19) of the adjoint system:

$$\ddot{\vec{p}}_V = -\dot{\vec{p}}_r = \vec{p}_V \cdot \left[\overline{\overline{\nabla^T . \vec{g}^T(\vec{r}, t)}}\right] \quad \left(\nabla = \frac{\partial}{\partial x_i}, \text{ line operator}\right) \tag{45}$$

therefore:

$$\ddot{\vec{\rho}} = \vec{\gamma} + \vec{g}\left(\vec{r}, t\right) + \left[\overline{\overline{\nabla^T . \vec{g}^T(\vec{r}, t)}}\right]\left(\vec{\rho} - \vec{r}\right) \tag{46}$$

or, finally,

$$\ddot{\vec{\rho}} = \vec{\gamma} + \vec{g}\left(\vec{\rho}, t\right) + o\left(|\vec{p}_V|\right). \tag{47}$$

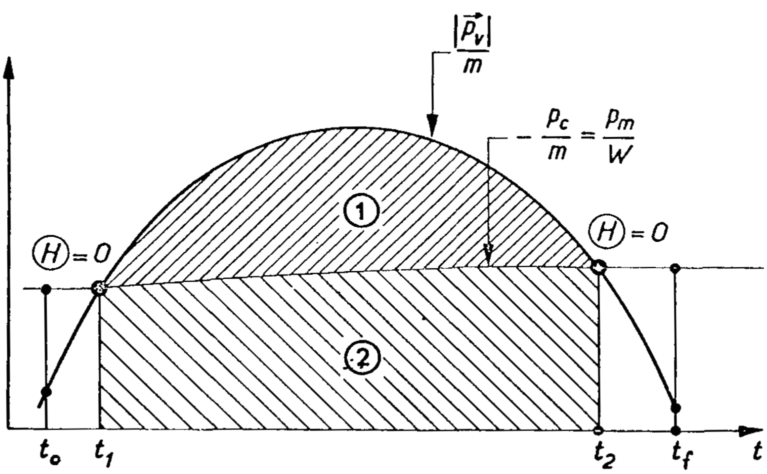


Fig. 5. Optimal Length of the Maximal Thrust Arcs.

Therefore at an order of $|\vec{p}_V|$ approximately higher than the first, the point P *shifts as if it were subjected at every instance to the same weight field and to the same thrust acceleration as the mobile* M (the weight obviously being calculated for P and not for M).

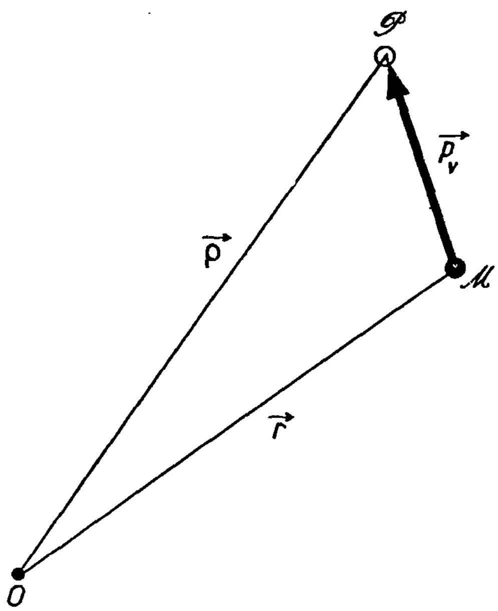


Fig. 6. Efficiency Vector.

Since the adjoint is defined only by about 1 multiplicative factor, fixed by the normalization condition, it is always possible to fix this normalization is such a way that $|\vec{p}_v| \ll |\vec{r}|$ and that the mentioned characteristic is precisely verified.

For convenience we formerly chose the normalization conditions:

$$p_{mf} \left(ou\ p_{Cf},\ ou\ p_{Jf} \right) = c_m \left(ou\ c_C\ ou\ c_J \right) = -1.$$

It is enough to take a modulus value $\ll 1$. /25

Therefore the optimal acceleration law can *theoretically* be defined in the following manner (Figure 7):

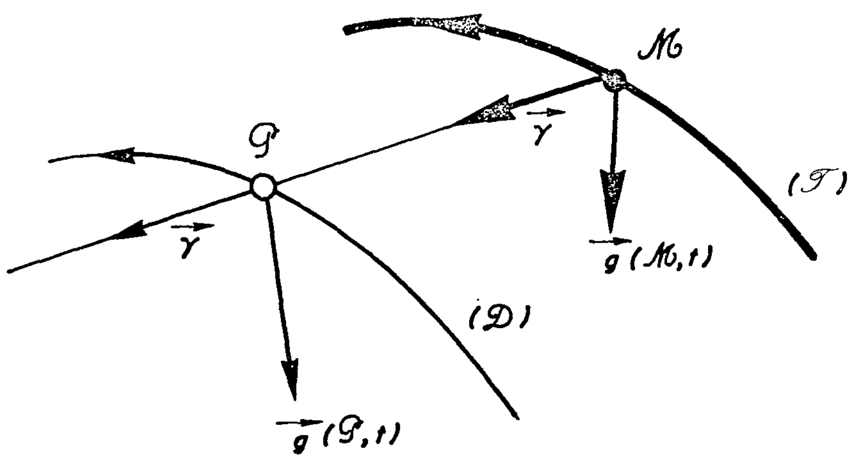


Fig. 7. Directrix Curve.

a pilot mobile *P* furnished with a propulsor is launched at the initial instant into the relative position $\vec{p}_{v_0}$ in relationship to M_0, close to M_0, and with the relatively low velocity $\dot{\vec{p}}_{v_0} = -\vec{p}_{r_0}$. The mobiles *M* and *P* are then subjected to the same acceleration $\vec{\gamma}$ directed according to $\vec{MP}$ [and proportional to *MP* for a propulsion system (S_2)]. While the mobile *M* describes the trajectory (*T*), the pilot mobile *P* describes the "*directrix curve*" (*D*) and furnishes the optimal thrust direction.

Instead of this elegantly theoretically solution there is an evident preference for the more traditional solution where the optimal acceleration law is remembered in the guide-pilot system of the mobile *M*, since at any rate the data of the initial conditions $\vec{p}_{r_o}$ and $\vec{p}_{V_o}$ (six parameters) presumes that these quantities have been previously calculated as a function of the mission to achieve, i.e. that the optimization problem has been resolved. Therefore the pilot mobile has only a purely theoretical usefulness!

I,2.3.5. Case of gravitational field independent of time.

In the case of a gravitational field independent of time, since the second members of the equations of movement (11) - (13) do not explicitly contain the variable t, it is possible to write the Hamiltonian first integral (see §I,1.5):

$$H^* = \vec{p_r}\cdot\vec{V} + \vec{p_V}\cdot\vec{g}(\vec{r}) + \frac{|\vec{p_V}|F^*}{m} - p_m q^* = C^{te} \tag{48}$$

which for a singular solution becomes $\left(|\vec{p_V}| \equiv \frac{m p_m}{W_{max}}\right)$

$$H^* = \vec{p_r}\cdot\vec{V} + \vec{p_V}\cdot\vec{g}(\vec{r}) = C^{te}. \tag{49}$$

On the other hand we have seen that it is then possible to define some generalized "orbits" and to consider some transfers without rendezvous. To say the final position M_f on the final "orbit" is not imposed comes down to saying, in particular, that it is equivalent to obtaining at the instant t_f the elements $\vec{r}_f$, $\vec{V}_f$ or the elements $\vec{r_f} + \vec{V_f}\,\delta t$ and $\vec{V_f} + \vec{g_f}\,\delta t$ relative to a point close to the final "orbit".

Since $\vec{p}_{r_f}$ and $\vec{p}_{V_f}$ are the coefficients of influence of the variations /26
$\delta\vec{r}_f$ and $\delta\vec{V}_f$, *compatible with the final restraints* (final imposed trajectory), on the index of performance S, this equivalence is translated by:

$$\left[\vec{p_r}\cdot(\vec{V}\delta t) + \vec{p_V}\cdot(\vec{g}\delta t)\right]_f = 0 \tag{50}$$

which is again to say:

$$\left[-\dot{\vec{p_V}}\cdot\vec{V} + \vec{p_V}\cdot\vec{g}\right]_f = 0. \tag{51}$$

Now since the field $\vec{g}(\vec{r})$ is derived from the potential $-U(\vec{r})$, its energy is particularly attached to one "orbit":

$$E = \frac{V^2}{2} - U(\vec{r}) = C^{te} = \frac{V_o^2}{2} - U(\vec{r_o}). \qquad (52)$$

Equation (51) shows that the energy of the generalized osculating "orbits" at the instant t_f respectively to the trajectory of point P [directrix curve (D)] and to that of M [trajectory (T)] are equal. As a matter of fact:

$$E_{\mathcal{P}} - E_{\mathcal{M}} = DE = D\left(\frac{V^2}{2} - U\right) = \vec{V}\cdot\overrightarrow{DV} - \overrightarrow{grad\,U}\cdot\overrightarrow{Dr} =$$
$$\vec{V}\cdot\left(\dot{\vec{\rho}} - \dot{\vec{r}}\right) - \overrightarrow{grad\,U}\cdot\left(\vec{\rho} - \vec{r}\right) = \vec{V}\cdot\dot{\overrightarrow{p_V}} - \vec{g}\cdot\overrightarrow{p_V} \qquad (53)$$

is really zero at instant t_f, because of (51).

1,2.4. Case of the Central Field

The hypothesis of the central field greatly simplifies the problem of optimal transfers by the fact that the natural movement is integrable. In order to benefit as much as possible from this integration, the elements of the Keplerian orbit osculating *to the trajectory are taken as components of state. Obviously this is a matter of choosing the elements most appropriate to the problem; in particular this choice should not be modified in the case of weak eccentricities. The variable of position must be particularly well chosen, for it must be substituted for time, as a variable of description, in order to facilitate the integrations. The optimal thrust law is easily interpreted: the thrust is directed towards a pilot mobile which describes a Keplerian orbit, the elements of which are perturbated.*

1,2.4.1. Osculating orbital elements.

Since the field of gravitation $\vec{g}$ is central, with a center O $\left(\vec{g} = -\mu \frac{\vec{r}}{r^3}\right)$, the elements $\vec{r}$, $\vec{V}$ (Figure 1) at each instant t define the Keplerian orbit (O) "osculating" (in the sense of celestial mechanics) to the transfer trajectory (T) of the mobile M, i.e. the orbit which M would continue to describe if the thrust were suppressed beginning at this instant.

A transfer (with rendezvous) is therefore the passage of the mobile M, between the fixed instant t_0 and t_f, from a very determined position M_0 on the initial orbit (O_0) to a well-defined position M_f on the final orbit (O_f) with the correct velocity $\vec{V}_f$ (Figure 8). In a transfer without rendezvous, the position M_f on (O_f) is not imposed. The problem of interception [position rendezvous with a target mobile M_c describing the orbit (O_f) without the correct velocity being imposed] will not be studied.

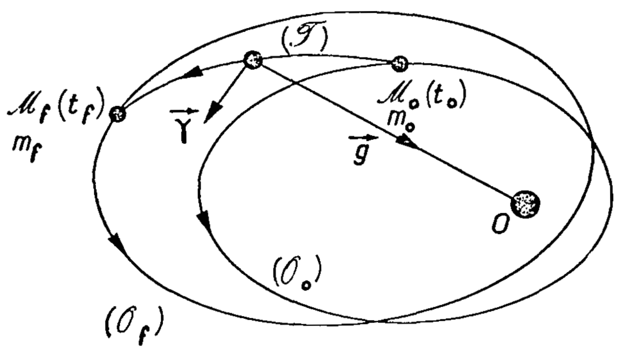


Fig. 8. Transfer. Central Field

The state $\vec{x}$ at the instant t of the mobile M can therefore be defined not only by the data of the elements $\vec{r}$, $\vec{V}$ and mass m, but also by the data of the osculating orbit (O) (five parameters), of the position of M on this orbit (one parameter), and of the mass m (or a performance index as a monotonous function of the mass m) (one parameter). /27

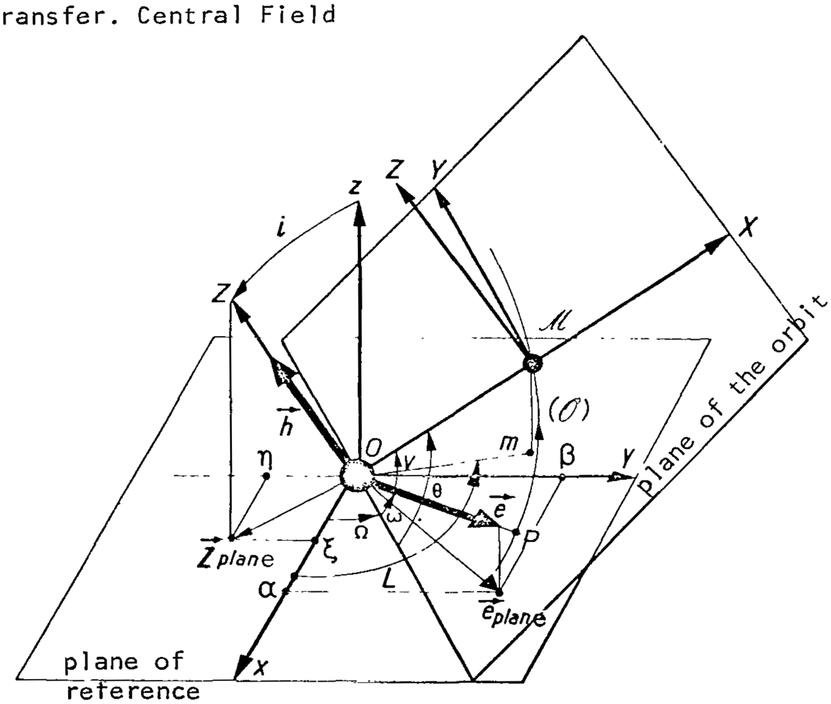


Fig. 9. Notations.

The osculating orbit (O) can be defined (Figure 9) by the datum:

1. of the unitary vector $\vec{Z} = \vec{h}/h$, perpendicular to the plane of (O), or better of the projection $\vec{Z}_{plane}$ of $\vec{Z}$ on the plane of reference Oxy, having as

components:

$$\overrightarrow{Z_{plane}} \begin{cases} \xi = \sin i \sin \Omega \\ \eta = -\sin i \cos \Omega \end{cases} \quad \text{(two parameters)} \tag{54}$$

2. of its semi-major axis a (one parameter)

3. of the "*perigee vector*": /28

$$\vec{e} = \frac{1}{\mu} \vec{V} \wedge \vec{h} - \frac{\vec{r}}{r} \quad \text{(Laplace's first integral, see Appendix 2)} \tag{55}$$

directed toward the perigee and of length e (eccentricity), or rather of the projection $\vec{e}_{plane}$ of $\vec{e}$ on the reference plane Oxy, having as components:

$$\vec{e}_{plane} \begin{cases} \alpha \\ \beta \end{cases} \quad \text{(two parameters)} \tag{56}$$

Note: The preceding choice is preferable to that which would consist of defining the osculating orbit *(O)* by the datum of $\vec{h}$ (kinetic moment: three parameters) and $\vec{e}$ (perigee vector: three parameters), connected by $\vec{h} \cdot \vec{e} = 0$ (one equation), because the semi-major axis a, connected to the period T by the equations

$$n^2 a^3 = \mu, \quad n = \frac{2\pi}{T} = \text{average movement}, \tag{57}$$

plays a basic role in the rendezvous, while the magnitude h of the kinetic moment, connected to the "parameter" $p = a\ (1-e^2)$ of the orbit by $h^2 = \mu p$ is of less interest.

On the other hand the position of the mobile *M* on the osculating orbit *(O)* can be defined by the datum of its straight (geocentric) ascent or of its ecliptic (heliocentric) longitude:

$$L = \left(\overrightarrow{Ox}, \overrightarrow{Om}\right). \tag{58}$$

It is preferable to take L as a parameter instead of θ (argument), *v* (true anomaly), u (eccentric anomaly), $M = u - e \sin u$ (average anomaly) or even, instead of the longitudes (in the sense of celestial mechanics):

$$\Omega + \theta \quad \text{or} \quad \Omega + \omega + M$$

because L varies monotonously as time (at least if i does not become equal to 90°, which will never be the case for close transfers) while it cannot be exact for the other quantities.

In particular, at the time of the impulsional thrust application, the position of the mobile M (therefore of its longitude L) does not vary, while the other quantities may have any variation in sign.

The use of the orbital elements is generally practical for treating transfer problems.

As a matter of fact, while the elements $\vec{r}$, $\vec{V}$ vary in large proportions along the entire length of the transfer trajectory (T), even on ballistic arcs, the five first orbital elements are constant on the ballistic arc and generally vary more continuously on the propulsed arcs, especially if it is a question of a transfer between close orbits.

On the other hand final conditions are generally expressed more simply, for in the majority of cases they directly concern the orbital elements rather than the elements $\vec{r}$, $\vec{V}$ [fixed semi-major axis (or energy), orientation of the plane of the fixed orbit, etc. ...]. In particular the condition of non-rendezvous, that is to say the final position M_f on the indifferent final orbit (O_f), is shown very simply (indifferent final straight ascent L_f).

Finally the choice of orbital elements associated with the linearization hypothesis leads to a constant adjoint for transfers between close orbits.

1,2.4.2. Variation of orbital elements.

The variation of the orbital elements is given by the *perturbation formulae:*

$$\dot{\vec{Z}}_{\text{plane}} = \frac{1}{h}\left(\vec{r} \wedge \vec{\gamma}\right)_{\text{plane}} \tag{59}$$

$$\dot{a} = \frac{2a^2}{\mu}\left(\vec{V} \cdot \vec{\gamma}\right)_{\text{plane}} \tag{60}$$ /29

$$\dot{\vec{e}}_{\text{plane}} = \frac{1}{\mu}\left[\vec{\gamma} \wedge \vec{h} + \vec{V} \wedge \left(\vec{r} \wedge \vec{\gamma}\right)\right]_{\text{plane}} \quad \text{(See Appendix 2)} \tag{61}$$

and the equation

$$\dot{L} = \frac{h \cos i}{r^2\left(1 - \sin^2 i \, \sin^2 \theta\right)}. \tag{62}$$

In the equations $\vec{r}$, $\vec{V}$, i, $\vec{h}$, θ are supposed to be expressed as a function of $\vec{Z}_{plan}$, a, $\vec{e}_{plan}$ and L.

I,2.4.3. Variations of the elements of the directrix orbit.

When the thrust acceleration $\vec{\gamma}$ is zero, the pilot mobile P, subjected uniquely to the central Newtonian field $\vec{g}(\vec{r}) = -\mu\,\vec{r}/r^3$, *describes a Keplerian directrix orbit* (*D*). When the thrust acceleration $\vec{\gamma}$ is not zero, the pilot mobile is subjected equally to the acceleration $\vec{\gamma}$ and it is possible to define at every instant the Keplerian directrix orbit (O_P) osculating to to the trajectory (*D*) of *P*, just as it has been possible to define the Keplerian orbit (*O*) osculating to the trajectory (*T*) of *M*. We propose to calculate the five first elements of this osculating orbit and their variations as a function of $\vec{\gamma}$.

If D designates the difference between one element of (O_P) and the corresponding element of (*O*):

$$\overrightarrow{Dh} = \vec{h}_{\mathcal{P}} - \vec{h} = D(\vec{r}\wedge\vec{V}) = D\vec{r}\wedge\vec{V} + \vec{r}\wedge D\vec{V} = \tag{63}$$

$$\vec{p_V}\wedge\vec{V} + \vec{r}\wedge\dot{\vec{p_V}} = \vec{p_V}\wedge\vec{V} + \vec{p_r}\wedge\vec{r}$$

$$\overrightarrow{De} = \vec{e}_{\mathcal{P}} - \vec{e} = D\left(\frac{1}{\mu}\vec{V}\wedge\vec{h} - \frac{\vec{r}}{r}\right) = \frac{1}{\mu}D\vec{V}\wedge\vec{h} + \frac{1}{\mu}\vec{V}\wedge\overrightarrow{Dh} - D\left(\frac{\vec{r}}{r}\right) = \tag{64}$$

$$\frac{1}{\mu}\vec{h}\wedge\vec{p_r} + \frac{1}{\mu}\vec{V}\wedge\overrightarrow{Dh} - \frac{\vec{p_V}}{r} + \frac{\vec{r}}{r^3}(\vec{r}\cdot\vec{p_V})$$

$$DE = E_{\mathcal{P}} - E = D\left(\frac{\vec{V}^2}{2} - \frac{\mu}{r}\right) = \vec{V}\cdot D\vec{V} + \mu\,\frac{\vec{r}\cdot\overrightarrow{Dr}}{r^3} = -\vec{V}\cdot\vec{p_r} + \mu\,\frac{\vec{r}\cdot\vec{p_V}}{r^3}\,. \tag{65}$$

If the *optimal* thrust acceleration $\vec{\gamma}^* = \gamma^*\vec{p_V}/|\vec{p_V}|$ is applied, the variations of these differences are given by:

$$\frac{d\overrightarrow{Dh}}{dt} = D\frac{d\vec{h}}{dt} = D(\vec{r}\wedge\vec{\gamma}^*) = D\vec{r}\wedge\vec{\gamma}^* = \vec{p_V}\wedge\vec{\gamma}^* = 0 \tag{66}$$

$$\frac{d\vec{De}}{dt} = D\frac{d\vec{e}}{dt} = \frac{1}{\mu} D\left[\vec{\gamma}^* \wedge \vec{h} + \vec{V} \wedge (\vec{r} \wedge \vec{\gamma}^*)\right] = \frac{1}{\mu}\left[\vec{\gamma}^* \wedge \vec{Dh} - \vec{p_r} \wedge (\vec{r} \wedge \vec{\gamma}^*)\right] \tag{67}$$

$$\frac{dDE}{dt} = D\frac{dE}{dt} = D(\vec{V} \cdot \vec{\gamma}^*) = \dot{\vec{p_V}} \cdot \vec{\gamma}^* = \gamma^* \frac{d|\vec{p_V}|}{dt}. \tag{68}$$ /30

It is verified that if $\vec{\gamma} = 0$, $\vec{\vec{Dh}}$, $\vec{\vec{De}}$ and DE are constant. But furthermore, we see that $\vec{\vec{Dh}}$ is constant, even when the thrust acceleration is applied, which furnishes a first integral for the optimization problem:

$$\vec{Dh} = \vec{p_V} \wedge \vec{V} + \vec{p_r} \wedge \vec{r} \tag{69}$$
= constant vector when *optimal* $\vec{\gamma}^*$ is applied in a *central field*.

This first integral is pointed out, among other places, in [27]. Here it is connected to the notion of osculating directrix orbit.

The interpretation of $\vec{\vec{Dh}}$ is easy:

suppose that at any instant t, we give the elements $\vec{r}$, $\vec{V}$ a *joint rotation* $\vec{\vec{\delta R}}$. If this rotation is compatible with the final restraints, the variation in the performance index S is then:

$$\vec{p_r} \cdot \vec{\delta r} + \vec{p_V} \cdot \vec{\delta V} = \vec{p_r} \cdot (\vec{\delta \mathcal{R}} \wedge \vec{r}) + \vec{p_V} \cdot (\vec{\delta \mathcal{R}} \wedge \vec{V}) =$$

$$\vec{\delta \mathcal{R}} \cdot (\vec{r} \wedge \vec{p_r} + \vec{V} \wedge \vec{p_V}) = -\vec{\delta \mathcal{R}} \cdot \vec{Dh}$$

$-\vec{\vec{Dh}}$ then appears as the coefficient of influence $\vec{\vec{p}}_R$, on the performance index S, of a joint rotation $\vec{\vec{\delta R}}$ compatible with the final restraints. In particular, if the final kinematic state $\vec{\vec{r}}_f$, $\vec{\vec{V}}_f$ is defined approximately at a joint rotation $\delta\theta$ around axis $\vec{k}$, $\vec{\vec{p}}_R \cdot \vec{\vec{\delta R}}$ must be zero for every rotation $\vec{\vec{\delta R}} = \vec{k}\delta\theta$ around this axis, and therefore $\vec{\vec{p}}_R \cdot \vec{k} = 0$. Likewise, if the final kinematic state is defined approximately for one joint rotation, $\vec{\vec{p}}_R = 0$.

Finally, equation (68) shows that on a singular arc where $|\vec{\vec{p}}_V| \equiv C^{te}$

$$DE = -\vec{V} \cdot \vec{p_r} + \mu \frac{\vec{r}}{r^3} \cdot \vec{p_V} = C^{te} = -H^* \tag{70}$$

which is nothing but the Hamiltonian first integral (49).

On a singular arc the energy of the osculating directrix orbit (O_P) and the energy of the osculating orbit (*O*) differ by one constant.

These few generalities concerning optimal transfers emphasize the complexity of the problem. Even in the case of central gravitational field, the integration of the equations of movement in the attached equations (one times the expression of the optimal thrust carried in these equations) with the conditions at the borders of the two extremities is analytically impossible in general. It is necessary to have recourse to some numerical integrations [34, 35].

However the analytical study has been able to be pursued in particular cases: singular case [31-33], transfers of indifferent duration [4, 7-19] and finally transfers between close orbits.

We are going to begin this last case now.

I,3. TRANSFERS BETWEEN CLOSE ORBITS

The hypothesis of short differences from the osculating orbit to the transfer course, referring to a reference nominal orbit, permits the problem to be linearized. /31

The major simplification furnished by this hypothesis is to render the variation of the adjoint negligible.

The resolution of the optimization problem is then made in three steps:

1. *The adjoint being given a priori, (with a certain number of zero components, corresponding to the orbital elements of which the final values are indifferent) the optimal thrust is deduced from it.*

2. *This optimal thrust is carried into the linearized equations of movement which are integrated. From these are deduced the variations of the orbital elements and consumption as a function of the adjoint.*

3. *The adjoint is determined a posteriori beginning with variations in the elements imposed by the transfer (inversion) and the consumption is deduced from it, as is the law of optimal thrust.*

I,3.1. Linearization of the Problem.

Our goal is to describe the linearized equations of movement (variation of the orbit elements, consumption),

1. With a description variable angular to position $\bar{u}$ instead of time t (which becomes a state component) in order to permit an easier ulterior integration,

2. So that the second members of these equations do not contain the state, which assures a constant adjoint,

3. So as to be able to treat the problems of *distant rendezvous on close orbits* and not only problems of close rendezvous, as is the case in the majority of studies [53, 54].

For this, the sixth element t is replaced by a parameter τ of which the variation $\Delta\tau$ is easily interpreted.

I,3.1.1. Nominal orbit.

Therefore in what follows we shall suppose that, *during the transfer*, the osculating orbit (O) is not far away from a nominal orbit $(\bar{O})$ (maximum separation $(\delta O) = (O) - (\bar{O})$ of the order of $\varepsilon \ll 1$).

Since the nominal orbit has to play a priveleged role, in order to simplify the calculations, we shall choose the following units:

unit of length: $\bar{a}$ = semi-major axis of $(\bar{O})$, (1)

unit of time: $\frac{\bar{T}}{2\pi}$ = [period on the orbit $(\bar{O})$]/2π. (2)

Then the unit of velocity is the circular velocity $V_{circ}(\bar{a})$ at the distance $\bar{a} = 1$ from the center of attraction and the unit of acceleration, the acceleration of gravitation $g(\bar{a})$ at the distance $\bar{a} = 1$ from the center of attraction.

We shall also choose (Figure 1): /32

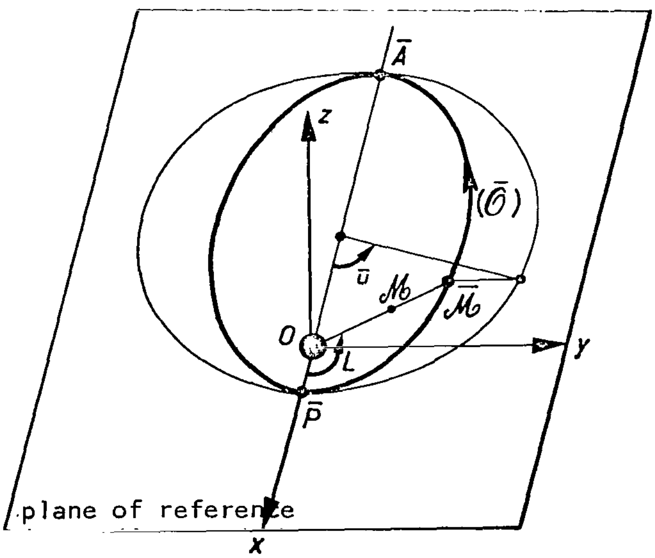


Fig. 1. Nominal Orbit.

$$\bar{i} = 0 \qquad (3)$$

and if $\bar{e} \neq 0$,

$$\bar{\Omega} + \bar{\omega} = \bar{\varpi} = \bar{\Lambda} \qquad (4)$$

(straight ascent of the perigee) = 0.

I,3.1.2. Change of description variable.

In order to carry out the integrations, it is suitable to change the description variable: instead of the time t, we shall take the parameter $\bar{u}$ defined by:

$$tg\,\frac{\bar{u}}{2} = \sqrt{\frac{1-\bar{e}}{1+\bar{e}}}\; tg\,\frac{L}{2} \qquad (5)$$

$\bar{u}$ *is the eccentric anomaly on the nominal orbit corresponding to the straight ascent of the real mobile.* $\bar{u}$ like L varies monotonously with the time t.

This description variable is more convenient than the variables below:

1. Eccentric anomaly u on the osculating orbit *(O)*:

   this anomaly fixes very well the real position of the mobile, but runs the risk of not varying monotonously with the time t, because it is sensitive to the movement of the perigee.

2. Eccentric anomaly of a fictitious mobile which describes the nominal orbit *of a Keplerian movement* as a function of time t:

   this anomaly can replace the variable t, but integration is facilitated only if it also carried into the second member of the equations of movement. This is possible only if error in position remains small (this anomaly must remain close to $\bar{u}$). Therefore it is impossible to treat some *distant* rendezvous problems.

Let us note that the nominal mobile $\bar{M}$ of straight ascent L describes the /33 nominal orbit *(O)* of a Keplerian movement as a function of the *nominal time* $\bar{t}$ such as:

$$\bar{r}^2 dL = \bar{a}\bar{b}\, d\bar{M} = \bar{a}\bar{b}\bar{n}\, d\bar{t} = \bar{r}\bar{b}\, d\bar{u}. \tag{6}$$

The description variable $\bar{u}$ now fixes the straight ascent L of the real mobile *M* on the osculating orbit *(O)*.

In the case of a rendezvous, it is also indispensable to know the time t which becomes a *state* variable. It equivalent to knowing t or:

$$\delta t(L) = t(L) - \bar{t}(L) \tag{7}$$

with the following conventions (fixing the origins of the time t and $\bar{t}$)

$$t(L_o) = t_o = 0 \tag{8}$$

$$\delta t(L_f) = t(L_f) - \bar{t}(L_f) = 0 \qquad \left[\text{fixed } \bar{t}(L_f) = t_f, \text{ therefore } \bar{t}(L_o)\right]. \tag{9}$$

Therefore *if the final orbit* (O_f) *is taken as the nominal orbit**, $\delta t(L)$ is nothing but the *temporal* lag (on the final orbit) of the nominal mobile $\overline{M}$ of the same straight ascent L as the real mobile M, in relation to the target mobile M_c (Figure 2).

These three mobiles $\overline{M}$, M and M_c coincide at the final instant t_f.

With the new variable introduced, the equations of movement are written:

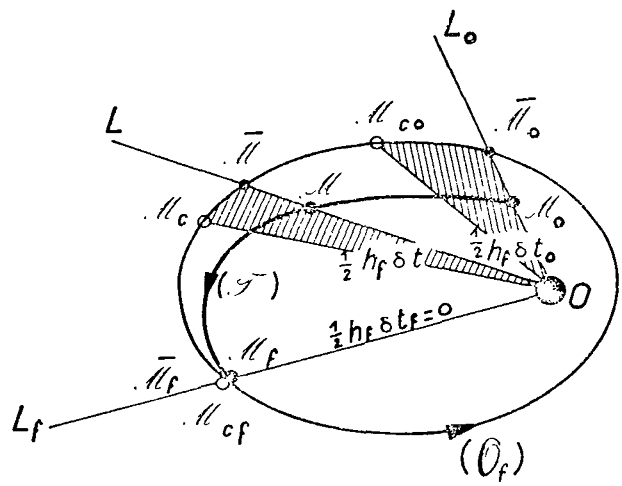


Fig. 2. Geometric Interpretation of δ2.

$$\vec{Z}'_{plane} = \frac{d\vec{Z}_{plane}}{d\bar{u}} = \frac{\bar{b}}{\bar{r}} \cdot \frac{r^2(1-\sin^2 i \sin^2\theta)}{h\cos i} \, \frac{(\vec{r}\wedge\vec{\gamma})_{plane}}{h} \tag{10}$$

*This hypothesis, which permits an easy interpretation of δt_o and $\Delta\tau$, is however *not at all obligatory*. Often it is even preferable to choose a nominal orbit intermediate between the initial and final orbits, in order to increase the exactitude of the calculations. This transposition is immediate.

$$a' = \frac{da}{d\bar{u}} = \frac{\bar{b}}{\bar{r}} \; \frac{r^2(1-\sin^2 i \sin^2\theta)}{h\cos i} \; 2a^2\, \vec{V}\cdot\vec{\gamma} \tag{11}$$

$$\vec{e}\,'_{plane} = \frac{d\vec{e}_{plane}}{d\bar{u}} = \frac{\bar{b}}{\bar{r}} \; \frac{r^2(1-\sin^2 i \sin^2\theta)}{h\cos i} \left[\vec{\gamma}\wedge\vec{h} + \vec{V}\wedge\left(\vec{r}\wedge\vec{\gamma}\right)\right]_{plane} \tag{12}$$

$$\delta t' = \frac{d(\delta t)}{d\bar{u}} = \frac{\bar{b}}{\bar{r}} \; \frac{r^2(1-\sin^2 i \sin^2\theta)}{h\cos i} - \frac{\bar{r}}{\bar{a}\bar{n}} \tag{13}$$

$$C' = \frac{dC}{d\bar{u}} = \frac{\bar{b}}{\bar{r}} \; \frac{r^2(1-\sin^2 i \sin^2\theta)}{h\cos i}\, \gamma \qquad (S_1) \tag{14}$$

$$J' = \frac{dJ}{d\bar{u}} = \frac{\bar{b}}{\bar{r}} \; \frac{r^2(1-\sin^2 i \sin^2\theta)}{h\cos i}\, \frac{\gamma^2}{2} \qquad (S_2) \tag{15}$$

$(\mu = 1)$.

These are the equations which must be linearized.

1,3.1.3 Infinitesimal transfer.

Let us be a little more precise about the notion of infinitesimal transfer.

In order to be able to linearize equations of movement, it is not enough to suppose that the initial orbit (O_0) and final orbit (O_f) are close. As we have done, §1,3.1.1, it is also necessary to suppose that, *during the transfer*, the distance between the osculating orbit (O) and these two orbits remains small [or rather that (O_0), (O_f) and (O) are close to the nominal orbit $(\bar{O})$].

Since catching up δt_0 is done by varying, during the transfer, the period of the osculating orbit, therefore by varying δa of its semi-major axis of which the maximal amplitude is of the order of $|\delta t_0|/\Delta t$, it is necessary to suppose that:

$$\varepsilon = \max\left(|\Delta\xi|,\ |\Delta\eta|,\ |\Delta a|,\ |\Delta\alpha|,\ |\Delta\beta|,\ \frac{|\delta t_0|}{\Delta t}\right. \tag{16}$$

The final term $|\delta t_0|/\Delta t$ entering into the definition of ε is essential, as the following example shows: to catch up to a target mobile which is on the same orbit as the pursuer, but with a temporal lead δt_0, the maximal seperation $|\delta a|$ during the rendezvous is of the order of $|\delta t_0|/\Delta t$, while the total variation Δ of the first five elements (and particularly Δa) is zero.

The characteristic velocity of the transfer:

$$\Delta C = \int_{t_0}^{t_f} \gamma \, dt \tag{17}$$

is of the order of ε.

For the propulsion systems (S_1), it is also necessary to suppose that the mass m of the mobile varies little (relative variation $|\Delta m|/\overline{m}$ of the order of ε); the nominal mass $\overline{m}$ is then taken as unity. This condition implies that the ejection velocity W is large enough for the nominal orbital velocity:

$$W \geqslant \frac{\Delta C}{O(\varepsilon)} = \overline{V}_A \tag{18}$$

For propulsion systems (S_2), this hypothesis is not necessary because only the thrust acceleration γ intervenes in the equations. However we shall do it sometimes simply to connect the variation of mass Δm to the variation in the performance index ΔJ:

$$\Delta J = P_{max}\left(\frac{1}{m_f} - \frac{1}{m_o}\right) \simeq P_{max} \frac{|\Delta m|}{\overline{m}^2} = P_{max} |\Delta m| \ si \ \overline{m} = 1 \tag{19}$$

1,3.1.4 Linearization.

If, in the second members of the equations of movement (10) - (15), the osculating orbit (O) is replaced by the nominal orbit $(\overline{O})$ and the mass m by the nominal mass $\overline{m} = 1$, the *relative* error committed in the solution of the problem is of the order of ε.

If ε is small enough, this error is permissible in so far as the variations of the elements of orbit (since these variations are at most of the order of ε, the absolute error is of the order of ε^2) are concerned. On the other hand it is unacceptable in rendezvous problems in determining the time t (or what comes to the same thing, the lag $\delta t(L)$. As matter of fact, equation (13) would furnish a constant lag. In equation (13) the osculating elements should be carried with a relative precision ε^2.

Designating the separation with straight ascent L between the real state and the nominal state by: /36

$$\delta x = x - \bar{x} \tag{20}$$

the *linearized* equations of movement are written (omitting except where necessary for comprehension a writing of the nominal sign $(\bar{\ })$ in the second members):

$$\vec{Z}'_{\text{plane}} = \delta\vec{Z}'_{\text{plane}} = \frac{r}{\sqrt{1-e^2}}\left(\vec{r}\wedge\vec{\gamma}\right)_{plane} \tag{21}$$

$$a' = \delta a' = 2r\,\vec{V}\cdot\vec{\gamma} \tag{22}$$

$$\vec{e}'_{\text{plane}} = \delta\vec{e}'_{\text{plane}} = r\left[\vec{\gamma}\wedge\vec{h} + \vec{V}\wedge\left(\vec{r}\wedge\vec{\gamma}\right)\right]_{plane} \tag{23}$$

$$\delta t' = \frac{3}{2}r\,\delta a - \frac{2r}{1-e^2}\left(\frac{3}{2}\vec{e}+\vec{r}\right)\cdot\delta\vec{e} \tag{24}$$

$$C' = \delta C' = r\gamma \qquad (S_1) \tag{25}$$

$$J' = \delta J' = r\frac{\gamma^2}{2} \qquad (S_2) \tag{26}$$

with $r = 1 - e\cos u$, $h = \sqrt{1-e^2}$.

Noting that:

$$r\,\delta a = (1-e\cos u)\,\delta a = \frac{d}{du}\left[(u-e\sin u)\,\delta a\right] - (u-e\sin u)\,\frac{d\delta a}{du} \tag{27}$$

and

$$-\frac{2r}{1-e^2}\left(\frac{3}{2}\vec{e}+\vec{r}\right)\cdot\delta\vec{e} = \frac{d}{du}\left(\vec{\mathcal{R}}\cdot\delta\vec{e}\right) - \vec{\mathcal{R}}\cdot\frac{d\delta\vec{e}}{du} \tag{28}$$

where

$$\vec{\mathcal{R}} = -\frac{2}{1-e^2}\int\left(\frac{3}{2}\vec{e}+\vec{r}\right)r\,du = -\frac{2}{1-e^2}\left\{\left[\left(1-\frac{e^2}{2}\right)\sin u - \frac{e\sin 2u}{4}\right]\vec{x}\right. \tag{29}$$

$$\left. + \sqrt{1-e^2}\left[\frac{3e}{4} - \cos u + \frac{e\cos 2u}{4}\right]\vec{y}\right\} = \frac{r}{\sqrt{1-e^2}}\vec{Y} + \frac{r^2}{1-e^2}\vec{V}$$

is a periodic vector in u of period 2π, equation (24) can be replaced by: /37

$$\boxed{\tau' = \frac{3}{2}M\delta a' + \vec{\mathcal{R}}.\vec{\delta e}'} \tag{30}$$

where

$$\tau = -\delta t + \frac{3}{2}M\delta a + \vec{\mathcal{R}}.\vec{\delta e} \tag{31}$$

is a new parameter which can replace the time t or even the lag δt in a rendezvous problem.

As a matter of fact, with the conventions adopted above:

$$\tau_f = \delta t_f = 0 \tag{32}$$

because $\delta a_f = 0$ and $\vec{\delta e}_f = 0$, since we have chosen to make the nominal orbit coincide with the final orbit and

$$\tau_o = -\delta t_o + \frac{3}{2}M_o\,\delta a_o + \vec{\mathcal{R}_o}.\vec{\delta e_o}$$

$$\Delta\tau = \tau_f - \tau_o = \delta t_o + \frac{3}{2}M_o\,\Delta a + \vec{\mathcal{R}_o}.\vec{\Delta e}\,. \tag{33}$$

Fixing $\Delta\tau$ comes down to fixing δt_o (if Δa and $\vec{\Delta e}$ are fixed, which is the case in a rendezvous). From now on we shall therefore use:

$\Delta\tau$ = (initial temporal lag on the final orbit of the nominal mobile $\overline{M}$ of the same longitude L_o as the real mobile M_o, referred to the target mobile M_{co}) + $\frac{3}{2}M_o\Delta a$ + $\vec{R}_o$ - $\vec{\Delta e}$. (34)

The utilization of the parameter τ instead of time t is of interest because the physical interpretation of $\Delta\tau$ is simple, and especially, the second member of the equation in τ' no longer contains the state separation $\vec{\delta x}$, which simplifies the optimization problem (constant adjoint).

Let us call $\vec{X}$ the kinematic condition:

$$\vec{\mathcal{X}} = \begin{bmatrix} \vec{Z}_{plane} \\ a \\ \vec{e}_{plane} \\ \tau \end{bmatrix} \qquad (35)$$

and $\vec{\delta X}$ the distance between the kinematic state and the nominal kinematic state. Equations (21) - (23) and (30) can be written in the following vectorial form:

$$\vec{\delta \mathcal{X}}' = r \bar{\bar{K}} \vec{\gamma} \qquad (36)$$

where $\bar{\bar{K}}$ is the tensor of influence of the thrust acceleration $\vec{\gamma}$ on the kinematic state, a tensor which depends only on the position u (and obviously on e).

If $\vec{\gamma}$ has the following components:

$$\vec{\gamma} \begin{cases} \gamma_X \\ \gamma_Y \\ \gamma_Z \end{cases} \qquad (37)$$

in the mobile axes $MXYZ$ and if $\vec{Z}_{plane}$ and $\vec{e}_{plane}$ are defined by their /38
components in the fixed axes Oxyz, the rectangular matrix (6 × 3) representative of the tensor $\bar{\bar{K}}$ is:

$$K = \begin{bmatrix} 0 & 0 & K_{\xi Z} \\ 0 & 0 & K_{\eta Z} \\ K_{aX} & K_{aY} & 0 \\ K_{\alpha X} & K_{\alpha Y} & 0 \\ K_{\beta X} & K_{\beta Y} & 0 \\ K_{\tau X} & K_{\tau Y} & 0 \end{bmatrix} \qquad (38)$$

The elements of this matrix are given in Appendix 3.

It can be noticed that the elements of the first five lines of the matrix rk are polynomials in sin u and cos u and that u only occurs in the elements of the sixth line (term in u sin u and $rK_{\tau x}$ and term in u in $rK_{\tau y}$).

Definition. We shall call *near-circular transfers*, the transfers between orbits with eccentricity e of the same order as the magnitude ε of the transfer or smaller ($e \leq$ order ε). It is then possible to choose a circular nominal orbit ($\bar{e} = 0$), since that only introduces into the solution a relative error of the order $e \leq$ order ε.

We shall call *transfers between ellipses of weak eccentricity* transfers between orbits of which the eccentricity e is small in comparison to unity, but large in comparison to the magnitude ε of the transfer ($\varepsilon \ll e \ll 1$). Then the nominal orbit will be an ellipse of weak eccentricity ($\varepsilon \ll \bar{e} \ll 1$).

I,3.1.5. Transfers and Rendezvous

• The previous choice of variables permits the treatment, not only of rendezvous problems where Δu and $\Delta\tau$ are fixed, but also of transfer problems (without rendezvous), for which *the transfer angle* ΔL *is fixed* (fixed Δu) and the duration Δt is indifferent (whence: $\Delta\tau$ is indifferent).

On the other hand it is impossible to treat directly the problems of transfer (without rendezvous) for which the *duration* Δt *is fixed* and the angle of transfer ΔL indefinite (whence: Δu is indefinite). As a matter of fact it is not equivalent to fix Δt or to fix $\Delta\tau$ because $\bar{t}\,(L_f) = t_f$ being given, $\delta t_o =$ $= -\bar{t}(L_o)$ depends on the length of the Keplerian arc $\overset{\frown}{\bar{M}_o\,\bar{M}_f}$ on $(\mathcal{O}_f)$ therefore of L_f. At any rate the preceding reasoning presumed that the final orbit was known (within about one plane rotation, because only δa and $\vec{\delta e}$ intervene in τ). This cannot be the case in certain transfers.

The study of the transfers (without rendezvous) of *fixed duration* Δt, which we are leaving to the side, could at any rate be carried on by considering the solution as the optimal case of rendezvous when, with fixed Δt, we sweep away the values of the parameter Δu and of the elements of the orbit which are not imposed.

There is reason to think that the relative solution for such a problem, where the final optimal position u_{1f} corresponds to the value t_{1f} (fixed), differs little (generally) from the solution of the problem of the same transfer to final fixed position u_{1f}. In particular, the value t_{2f} of the final optimal time obtained in this last case should be little different from t_{1f}. Still, in the case of a large number of revolutions, it is possible that the two solutions may present significant differences.

I,3.2. Determination of the Optimal Thrust as a Function of the Adjoint.

The Maximum Principle is applied directly to the linearized problem in order to treat the particular case of transfers between close orbits independently of the more general results already obtained in Chapter I,2. However the direct connection is made in Appendix 4, by using the notion of canonical change of variables [25, 30].

The law of optimal thrust is easily interpreted thanks to the notions of "directrix orbit" (O_P) *and especially of the "efficiency curve"* (P).

For propulsion systems (S_1) *some general results are obtained concerning* /39
the maximum number of maximal thrust arcs per revolution and the singular cases (of the linearized problem).

I,3.2.1. Direct application of the Maximum Principle to the linearized problem.

If:

$$\vec{P} = \left[\vec{p_z}, p_a, \vec{p_e}, p_\tau\right] = \left[p_\xi, p_\eta, p_a, p_\alpha, p_\beta, p_\tau\right] \tag{39}$$

designates the "*kinematic adjoint*" (associated with the kinematic distance $\vec{\delta X}$), where $\vec{p}_z$ and $\vec{p}_e$ are two vectors *in the plane* Oxy (therefore with *two* components), the Hamiltonian is written:

$$\mathcal{H} = \vec{p}\cdot\vec{\delta x}' = r\,\vec{P}\overline{\overline{K}}\vec{\gamma} + \begin{cases} p_c\,\delta C'\ (S_1) \\ p_J\,\delta J'\ (S_2) \end{cases} = \vec{p_z}\cdot\vec{\delta z}' + p_{a_1}\delta a' + \vec{p_{e_1}}\cdot\vec{\delta e}' +$$

$$\begin{cases} p_c\,\delta C'\ (S_1) \\ p_J\,\delta J'\ (S_2) \end{cases} = r\left[\vec{p_V}\cdot\vec{\gamma} + \begin{cases} p_c\,\gamma\ (S_1) \\ p_J\,\dfrac{\gamma^2}{2}\ (S_2) \end{cases}\right] \tag{40}$$

where:

$$\boxed{\begin{aligned} \vec{p_V} &= \vec{P}\overline{\overline{K}} = 2p_{a_1}\vec{V} + \vec{h}\wedge\vec{p_{e_1}} + \left(\vec{p_{e_1}}\wedge\vec{V} + \frac{\vec{p_z}}{h}\right)\wedge\vec{r} \\ &= 2p_a\vec{V} + \vec{h}\wedge\vec{p_e} + \left(\vec{p_e}\wedge\vec{V} + \frac{\vec{p_z}}{h}\right)\wedge\vec{r} + p_\tau\left(3M\vec{V} - 2\vec{r}\right) \end{aligned}} \tag{41}$$

and

$$p_{a_1} = p_a + \frac{3M}{2}p_\tau \tag{42}$$

$$\vec{p_{e_1}} = \vec{p_e} + \vec{\mathcal{R}}\,p_\tau \tag{43}$$

H is maximum in reference to $\vec{\gamma}$ for:

$$\left(\frac{\vec{\gamma}}{\gamma}=\right)\vec{D}=\frac{\vec{p_V}}{|\vec{p_V}|} \tag{44}$$

and:

(S_1): $\gamma = \gamma_{max}\ U(\Theta)$ (U = measure of unity, $\Theta = |\vec{\vec{p}}_V| + p_c$ = commutation function) (45)

(S_2): $\gamma = |\vec{p_V}|$ (i.e. finally $\vec{\gamma} = \vec{\vec{p}}_V$) (46)

With a relative error ε, the vector $\vec{\vec{p}}_V$ defined in (41) is nothing but the "vector of efficiency" found in Chapter I,2., i.e. the adjoint vector attached to velocity $\vec{V}$. This point is also evident if we carry out the canonical change /40 of variables

$$(\vec{r},\vec{V},t) \Longrightarrow \begin{cases}(\vec{\mathcal{X}}, u) \\ \quad ou \\ (\vec{Z}_{plane}\, a, \vec{e}_{plane}, \tau, u)\end{cases}$$

(See Appendix 4).

Since the Hamiltonian H does not contain any separation δ, the adjoint $\vec{p}$ is constant.

I,3.2.2. Directrix orbit.

As we have already pointed out in the case of any field of gravitation, the extremity P of the vector $\vec{\vec{MP}} = \vec{\vec{p}}_V$ describes, in the fixed axes Oxyz and in the first order in $|\vec{\vec{p}}_V|$, a directrix curve (D) obtained by subjecting the pilot mobile P to the same field of gravitation and to the same thrust acceleration as the mobile M.

If, in place of the real mobile M, the nominal mobile $\overline{M}$ (corresponding to the same straight ascent L) is used, a relative error of the order ε only is committed for $\vec{\vec{p}}_V$ as we have just seen.

Now, the nominal mobile $\overline{M}$ describes the nominal orbit $(\overline{O})$ of a Keplerian movement ($\vec{\gamma} = 0$) *as a function of time* t. Therefore the nominal pilot mobile $\overline{P}$ also describes a Keplerian orbit [nominal directrix orbit $(\overline{O}_P)$] of a Keplerian movement ($\vec{\gamma} = 0$) *as a function of time* $\overline{\tau}$ (Figure 3).

The datum of the kinematic adjoint $\vec{P}$ (constant) is equivalent to the datum of the nominal directrix orbit $(\bar{O}_P)$ (five elements: $\vec{Z}_{\text{plane}P}$, a_P , $\vec{e}_{\text{plane}P}$ and of the initial position $\bar{P}_0$ of the nominal pilot mobile on this orbit (one element), as the following equations, demonstrated in Appendix 5, illustrate:

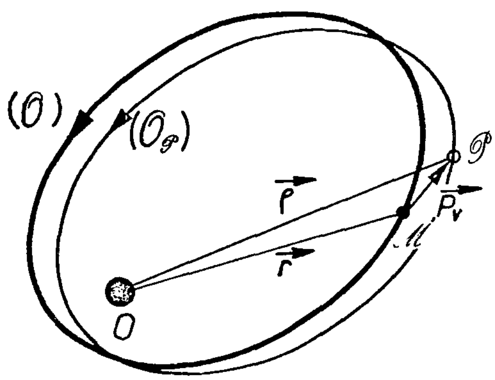


Fig. 3. Directrix Orbit.

$$\overrightarrow{Dh} = \vec{h}_{\mathcal{P}} - \vec{h} = \vec{p_Z} \wedge \vec{Z} + \vec{p_e} \wedge \vec{e} - p_\tau \vec{h} + \text{order } \vec{p_V}^2 \tag{47}$$

$$\left(= \vec{p_Z} \wedge \vec{Z} - p_\tau \vec{Z} + \text{order } \vec{p_V}^2 \text{ for } e = 0 \right)$$

$$\overrightarrow{De} = \vec{e}_{\mathcal{P}} - \vec{e} = \frac{\vec{p_Z}}{\sqrt{1-e^2}} \wedge \vec{e} + \sqrt{1-e^2}\, \vec{p_e} \wedge \vec{Z} + \text{order } \vec{p_V}^2 \tag{48}$$

$$\left(= \vec{p_e} \wedge \vec{Z} + \text{order } \vec{p_V}^2 \text{ for } e = 0 \right)$$

$$Da = a_{\mathcal{P}} - a = \frac{2}{h^2}\left(\vec{h} \cdot \overrightarrow{Dh} + \vec{e} \cdot \overrightarrow{De} \right) = -2 p_\tau + \text{order } \vec{p_V}^2 \tag{49}$$

(with $h = \sqrt{1 - e^2}$).

It is important to note that p_a does not intervene in these formulae. Therefore in the case where variation of the semi-major axis is the *only* variation imposed, i.e. where p_a is the only component of the adjoint not necessarily zero, the nominal directrix orbit coincides with the nominal orbit (tangential thrust).

On the other hand Da contains only the component p_τ of the adjoint. As a matter of fact, if there is no rendezvous ($p_\tau = 0$), we have demonstrated in § I,2.3.5., in the case of any gravitational field (but independently of time), that the energies of the "orbits" osculating at the final instant to the trajectories of the mobiles *M* and *P* were equal. Here the energies, and therefore the semi-major axes and the periods of the nominal directrix orbit and of the nominal orbit,are equal. The vector of efficiency $\vec{\vec{p}}_V \equiv \overrightarrow{\overrightarrow{MP}}$ is therefore a periodic vector in u of period 2π if there is no rendezvous (or more generally if $p_\tau = 0$). /41

On the other hand if $p_\tau \neq 0$ (which can only occur in the case of rendezvous), the periods of P and of M are different and the points M and P are separated more and more during the successive revolutions, and the vector $\vec{\vec{p}}_V \equiv \equiv \vec{\vec{MP}}$ rests more and more on the velocity $\vec{V}$ of M.

1,3.2.3. Efficiency curve.

The locus (O_P) of P in the fixed axes Oxyz is simple (conical of focus O) which permits us to follow easily the evolution of the direction (in absolute space) of the optimal acceleration and even the evolution of its magnitude in the case of propulsion systems (S_2).

However it is also convenient to consider the locus (P) of P in the system of axes of mobiles $MXYZ$ of origin M in order to follow the evolution of the direction of the optimal acceleration in axes connected this time to the mobile (which can be interesting from a practical point of view) and to determine the maximal thrust arcs and the ballistic arcs in the case of propulsion systems (S_1) (Figure 4).

Let us here recall that for a propulsion system (S_2) the optimal accelera- /42
tion in M is the vector $\vec{\gamma} = \vec{\vec{MP}} = \vec{\vec{p}}_V$ (if $p_j = -1$).

For a propulsion system (S_1), acceleration has $\vec{\vec{MP}} = \vec{\vec{p}}_V$ as a direction and as a modulus γ_{max} if $|\vec{\vec{p}}_V| > 1$, i.e. if P is outside of the sphere (Σ) with center M and radius 1, and 0 if $|\vec{\vec{p}}_V| < 1$, i.e. if P is inside the sphere (what has just been said presumes $p_c = -1$).

The commutation points $u_1, u_2 \ldots$ (separating the maximal thrust arcs from the ballistic arcs) are therefore the intersection points of the "*efficiency curve*" (P) and of the sphere (Σ).

When $p_\tau = 0$ (and in particular when there is no rendezvous), the components $\vec{\vec{p}}_{e_1} = \vec{\vec{p}}_e$ and $p_{a_1} = p_a$ are fixed in $\vec{\vec{p}}_V$ and as $\vec{r}$ and $\vec{V}$ are periodic in u, with a period of 2π, $\vec{\vec{p}}_V$ is periodic with the same period. The efficiency curve (P) is therefore a *closed* curve and the same acceleration law should be applied during successive revolutions. Figure 4a illustrates this case for a propulsion system (S_1).

On the other hand when $p_\tau \neq 0$ (which can occur only in the case of rendezvous), the components $\vec{\vec{p}}_{e_1}$ and p_{a_1} vary as a function of u, because of the terms M and $\vec{R}$, and $\vec{\vec{p}}_V$ contains a secular term (the presence of M and p_{a_1}). The efficiency curve (P) is no longer a closed curve. The acceleration law should be modified during successive revolutions. Figure 4b illustrates this case for a propulsion system (S_1).

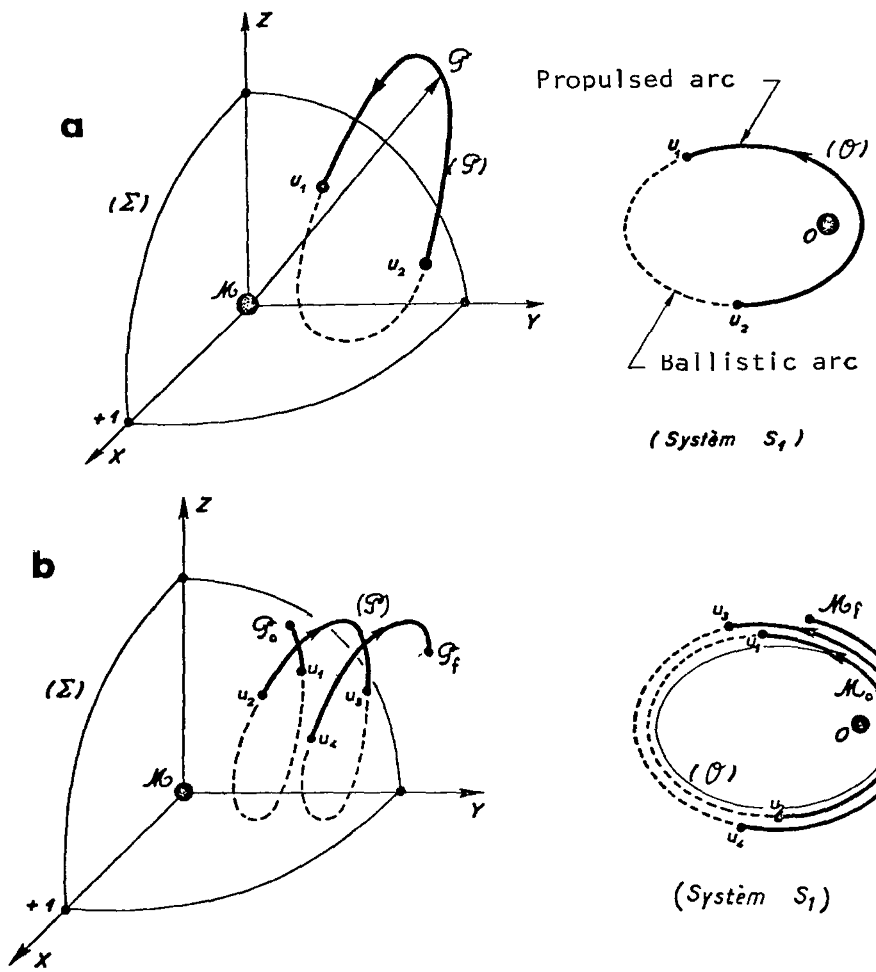


Fig. 4. Efficiency Curve.

a - Simple transfer, b - Rendezvous.

For $\Delta u >> 2\pi$, it is not $|p_\tau|$ but $\Delta u|p_\tau|$ which is comparable to the other components of the adjoint and thus:

$$\vec{p_V} \sim \left(2p_a + 3u p_\tau\right)\vec{V} + \vec{h} \wedge \vec{p_e} + \left(\vec{p_e} \wedge \vec{V} + \frac{\vec{p_z}}{h}\right) \wedge \vec{r}. \quad (50$$

1,3.2.4. Efficiency curve in the case of a circular nominal orbit (without rendezvous)

In the case where the nominal orbit (O) is circular (e = 0) (which is the case for transfers between near-circles), the vector of efficiency $\vec{p}_V$ is written (if there are no rendezvous, or better, if $p_T = 0$):

$$\vec{p_V} = 2p_a \vec{Y} + \vec{Z} \wedge \vec{p_e} + \left(\vec{p_e} \wedge \vec{Y} + \vec{p_Z}\right) \wedge \vec{X}. \tag{51}$$

Then positing (Figure 5):

$$\vec{p_e} = p_\alpha \vec{x} + p_\beta \vec{y} = p_e \left(\vec{x} \cos L_e + \vec{y} \sin L_e\right) \tag{52}$$

$$\vec{p_Z} = p_\xi \vec{x} + p_\eta \vec{y} = p_Z \left(\vec{x} \cos L_Z + \vec{y} \sin L_Z\right) \tag{53}$$

the components of $\vec{p}_V$ in the mobile axes are:

$$\vec{p_V} \begin{cases} X = p_\alpha \sin L - p_\beta \cos L = p_e \sin\left(L - L_e\right) \\ Y = 2p_a + 2p_\alpha \cos L + 2p_\beta \sin L = 2p_a + 2p_e \cos\left(L - L_e\right) \\ Z = p_\xi \sin L - p_\eta \cos L = p_Z \sin\left(L - L_Z\right). \end{cases} \tag{54}$$

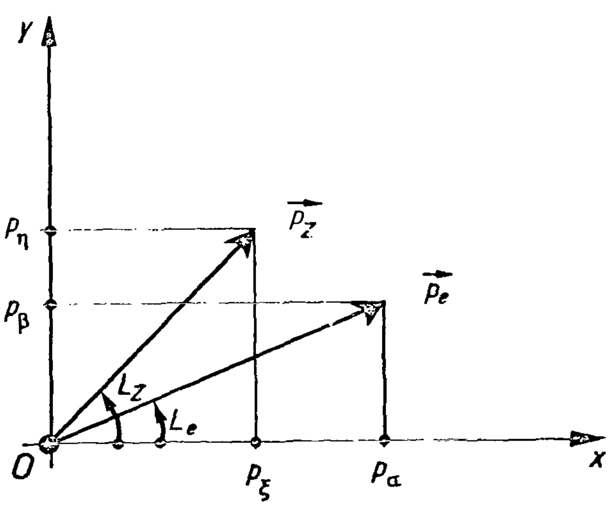


Fig. 5. Notations.

The efficiency curve (P) is an *ellipse* (Figure 6), a section of the elliptical cylinder (σ) of generatrices parallel to $\vec{MZ}$ and of equation: /43

$$(\sigma) \left| \; 4X^2 + \left(Y - 2p_a\right)^2 - 4p_e^2 = 0 \right. \tag{55}$$

and of the plane (π) of equation: /44

$$(\pi) \left| \; 2p_Z \cos\left(L_e - L_Z\right) X + p_Z \sin \right. \tag{56}$$

$$\left(L_e - L_Z\right)\left(Y - 2p_a\right) - 2p_e Z = 0.$$

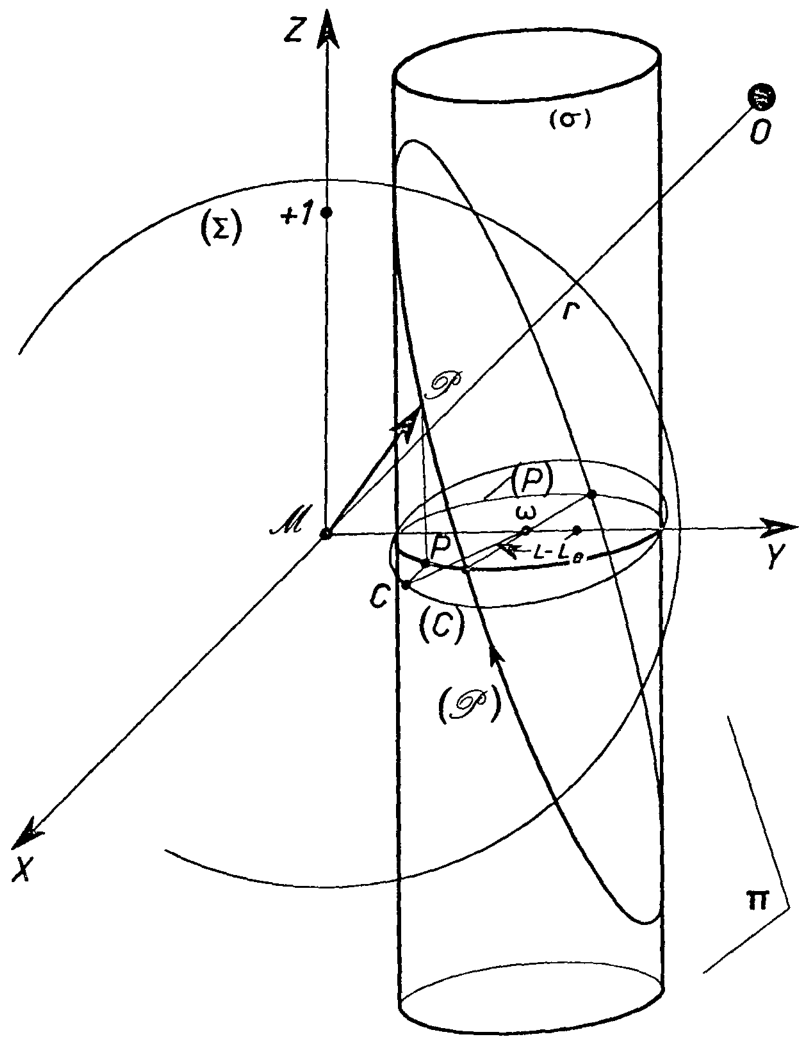


Fig. 6. Efficiency Curve for e = 0.

The plane (π) passes through point $\omega(X = 0, Y = 2p_a, Z = 0)$, the center of the elliptic base (P) of cylinder (σ) in the plane $Z = 0$, described by point P *of eccentric anomaly* $L - L_e$.

This result extends to the three-dimensional case Lawden's result [21] concerning plane transfers.

1,3.2.5. Propulsion systems (S_1) - Number of thrust arcs - Impulsional solutions.

Let us take the case where $p_\tau = 0$, i.e. where the efficiency curve (*P*) is a closed curve.

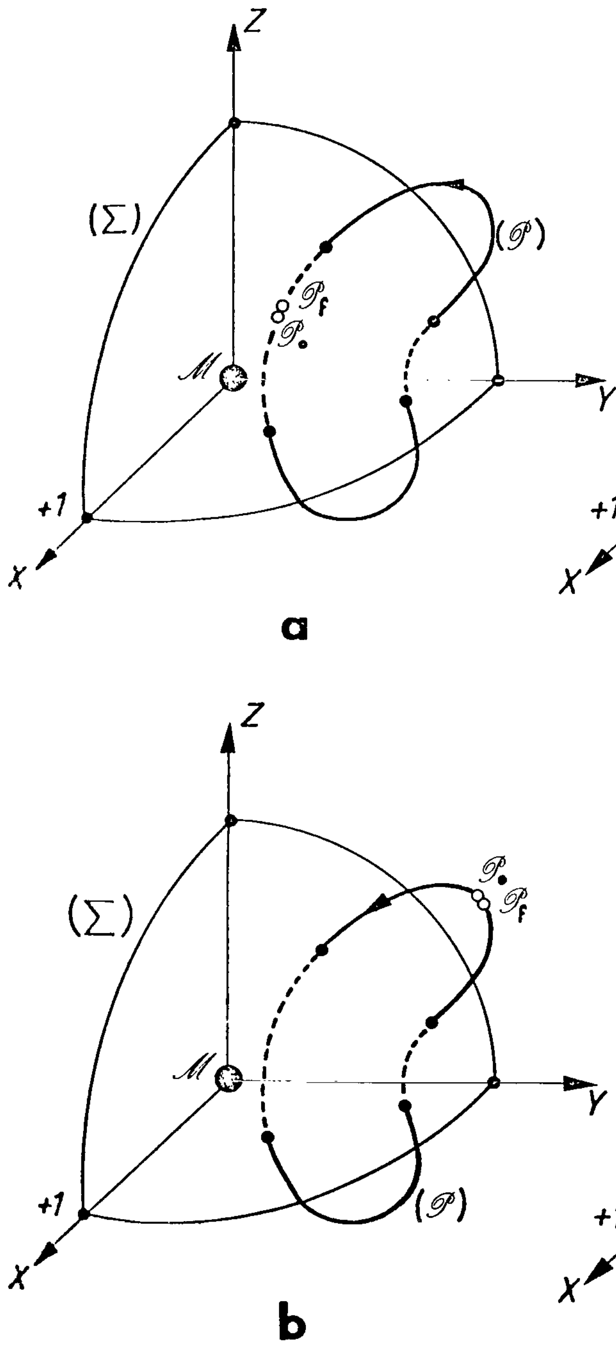


Fig. 7. Number of Maximal Thrust Arcs (Case of a Complete Number of Turns).

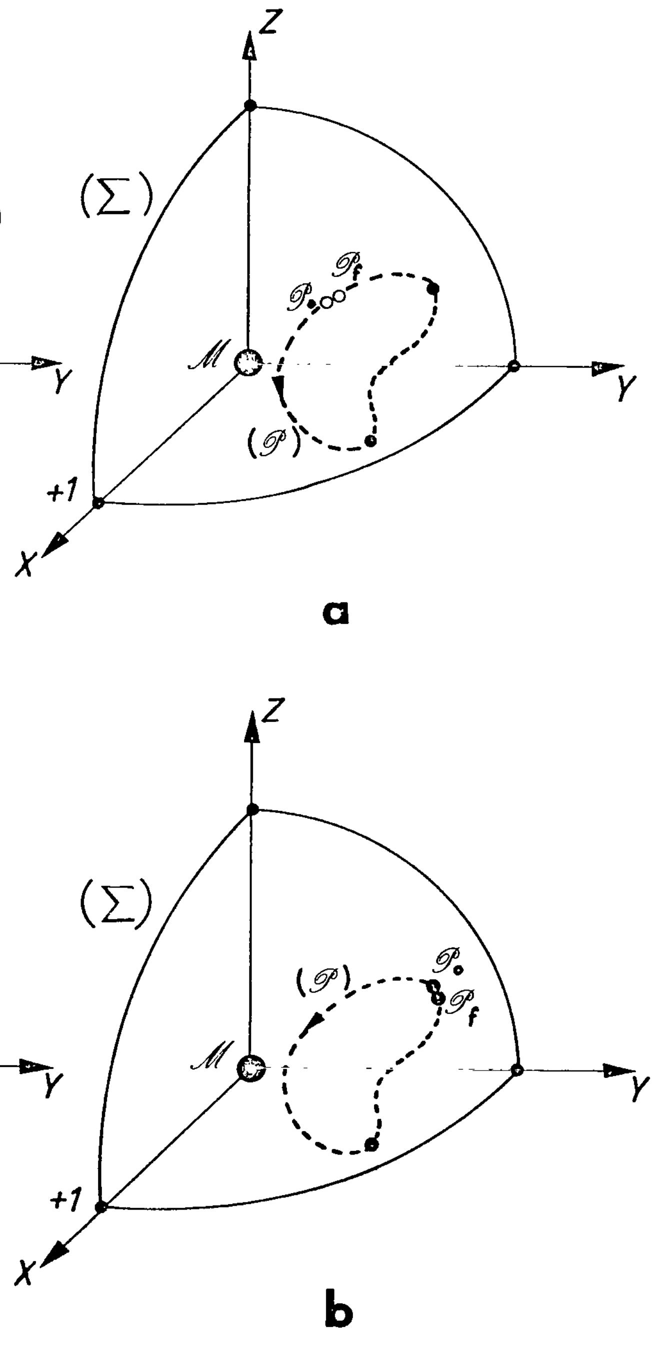


Fig. 8. Number of Impulses (Case of a Complete Number of Turns).

If the transfer duration corresponds to a complete number N of turns ($\Delta L = 2N\pi$), according to the initial position M_o of M on the orbit [or the initial position of P_o of P on the efficiency curve (P) (Figure 7)], the number n of maximal thrust arcs is equal to:

$$n = \begin{cases} Nn_1 \\ Nn_1 + 1 \end{cases}$$

where n_1 is the number of arcs of (P) outside the sphere (Σ).

This result extends to the impulsional case [(P) tangent to (Σ) in n_1 points] (Figure 8a and 8b). But we shall see that in this case the application of an impulse *in each* of the n points envisaged above is not always obligatory if $F_{max} = \infty$. (In the case of Figure 7b in particular, it is possible to reduce the impulses in P_o and P_f to a single impulse in P_o or P_f).

If now: $2N\pi < \Delta L < 2(N + 1)\pi$ and if the solution is not impulsional, we get: $Nn_1 \leq n \leq (N + 1)n_1 + 1$. If the solution is impulsional, it is necessary to make a distinction between transfers in less than one turn ($0 < \Delta L < 2\pi$) where (P) can present arcs outside of (Σ) (for example Figure 9) and transfers in more than one turn where (P) cannot present an arc exterior to (Σ) and where: $Nn_1 \leq n \leq (N + 1)n_1$.

The determination of n_1 is therefore basic. $2n_1$ is the number of roots in u of the equation:

$$\vec{P_V}^2 - 1 = PKK^T P^T - 1 = 0 \tag{57}$$

$(\ )^t$ = transposition.

The elements of the square symmetric matrix (6 × 6):

$$B = rKK^T = \begin{bmatrix} B_{\xi\xi} & B_{\xi\eta} & 0 & 0 & 0 & 0 \\ B_{\eta\xi} & B_{\eta\eta} & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{aa} & B_{a\alpha} & B_{a\beta} & B_{a\tau} \\ 0 & 0 & B_{\alpha a} & B_{\alpha\alpha} & B_{\alpha\beta} & B_{\alpha\tau} \\ 0 & 0 & B_{\beta a} & B_{\beta\alpha} & B_{\beta\beta} & B_{\beta\tau} \\ 0 & 0 & B_{\tau a} & B_{\tau\alpha} & B_{\tau\beta} & B_{\tau\tau} \end{bmatrix} \tag{58}$$

are given in Appendix 6. The elements of the minor relative to element $B_{\tau\tau}$ (which are the only ones which interest us here, for $p_\tau = 0$) are polynomials in sin u and cos u. More exactly the diagonal elements and *the element* $B_{a\alpha}$ are polynomials P(cos u) in cos u of degree 1 or 3. The other elements are of the form sin u Q(cos u), where Q(cos u) is a polynomial in cos u of degree 0 or 2.

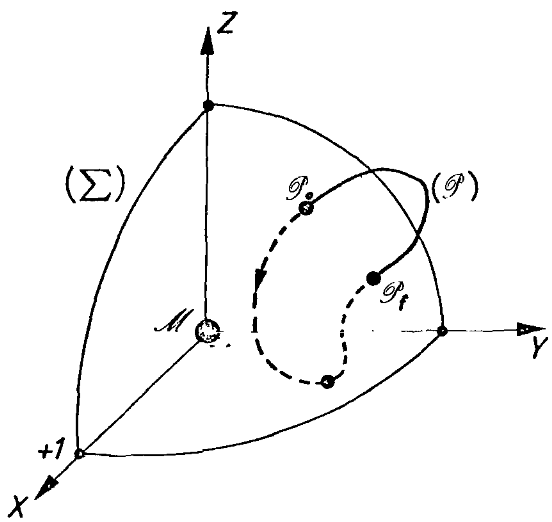


Fig. 9. Number of Impulses (Incomplete Number of Turns).

In the general case (p_ξ, p_η, p_a, p_α and p_β not zero), equation (57) is therefore written:

$$1 - e\cos u = PBP^T = P_3(\cos u) + \sin u\, Q_2(\cos u). \tag{59}$$

Or else:

$$P_3'(\cos u) = \sin u\, Q_2(\cos u) \tag{60}$$

and by raising it to the square:

$$P_6(\cos u) = \sin^2 u\, Q_4(\cos u) = Q_6(\cos u)$$

or finally:

$$P_6'(\cos u) = 0. \tag{61}$$

This equation admits a maximum of six roots in cos u. For each of these roots sin u is well determined by equation (60). This return to equation (60) eliminates the parasitic roots introduced by squaring.

Therefore there are at the maximum six commutation points, i.e. three arcs of (P) outside of (Σ). Therefore: $n_1 \leq n_{1max} \leq 3$ where n_{1max} is the maximum value of n_1 for all of the transfers (i.e. when e, p_ξ, p_η, p_a, p_α and p_β have any values at all). We cannot write $n_{1max} = 3$, for nothing proves *a priori* that the roots of (61) are real and that the inequalities $-1 \leq \cos u \leq +1$, $-1 \leq \sin u \leq +1$ are verified.

However we shall show in § II,5.3. that *there do exist*, at least for certain particular transfers between elliptical orbits of weak eccentricity ($\varepsilon << e << 1$), solutions to $n_1 = 3$ impulses per revolution. Therefore for all transfers between elliptical orbits ($\varepsilon << e < 1$), we can affirm that $n_{1max} = 3$, therefore that $n_1 \leq 3$.

On the other hand, for transfers between near-circles ($e \leq$ order $\varepsilon << 1$) where the nominal orbit is circular ($e = 0$), we have seen in the previous paragraph that the efficiency curve is an ellipse which can have only two arcs outside of sphere (Σ). Therefore there cannot be more than two thrust arcs per revolution and as solutions for two thrust arcs (or two impulses) are frequently found, we conclude from this that $n_1 \leq 2$ in this case. /47

Up to the present moment we have been reasoning in the most general case of transfer. The same reasoning applies to more particular classes of transfers.

Particular Transfers

The degrees of the polynomials P and Q in (59) depend on the *choice of the parameters* p_ξ, p_η, p_a, p_α and p_β *which have a non-zero value*. The variations of the corresponding orbital elements are imposed. The other variations are indifferent or, *exceptionally*, imposed. As a matter of fact a component of the adjoint can be zero while the variation of the corresponding element is imposed. For this it is enough for this imposed variation to coincide exactly with the induced variation which would be obtained in the problem where this variation would be considered indifferent.

Taking into consideration the exceptional character of this possibility, we shall associate with a choice of the parameters p_ξ, p_η, p_a, p_α and p_β which have a non-zero value, the class of transfers for which the variations of the corresponding orbital elements are imposed. Thus in general there are $2^5 - 1 = 31$ distinct classes of transfers.

In Appendix 7, the degrees of the polynomials P and Q, as well as an upper boundary for the maximum admissible number n_1 of maximal thrust arcs per revolution are given as a function of the class of transfers considered.

This upper limit is equal to three for most classes of transfers, except for the optimal variation of the semi-major axis Δa where it is equal to one and for the optimal rotation of the plane of the orbit ($\Delta\xi$, $\Delta\eta$) where it is equal to two (although the degree of P is equal to 3 and that of Q to 2, for P and Q contain $r = 1 - e \cos u \neq 0$ as a factor).

When $Q = 0$, equation (59) is an equation only in cos u. The commutation points and the maximal thrust arcs are symmetrical in relation to the major axis.

In the case of transfers between near-circular orbits ($e \leq$ order $\varepsilon << 1$), where the nominal orbit is circular (e = 0), we shall see in Chapter I,4. that the number of classes of transfers to be envisaged goes from 31 to 19 (see Appendix 7).

According to the class of transfers considered, $\vec{\vec{p}}_V^{\,2}$ can be a function of $\sin^2 L$ and $\cos^2 L$ (that is to say finally of sin L alone or cos L alone) which leads to a symmetry of the maximal thrust arcs in relatin to $\vec{\vec{Ox}}$ and to $\vec{\vec{Oy}}$, therefore in reference to zero or indeed as a function of sin L alone (symmetry in relation to $\vec{\vec{Oy}}$) or a function of cos L alone (symmetry in relation to $\vec{\vec{Ox}}$), or finally of sin L and cos L (no symmetry in relation to the axes $\vec{\vec{Ox}}$ and $\vec{\vec{Oy}}$).

If we exclude the singular cases (of the linearized problem) studied in the following paragraph, the admissible maximum number of maximal thrust arcs per revolution is equal to 2 except for the optimal variation of the semi-major axis where it is equal to 1 (see § II,1.2.).

I,3.2.6. Propulsion systems (S_1) - Singular cases (of the linearized problem)

In certain transfers (without rendezvous), it can happen that the efficiency curve *(P) is entirely situated on the sphere* (Σ) (Figure 10).

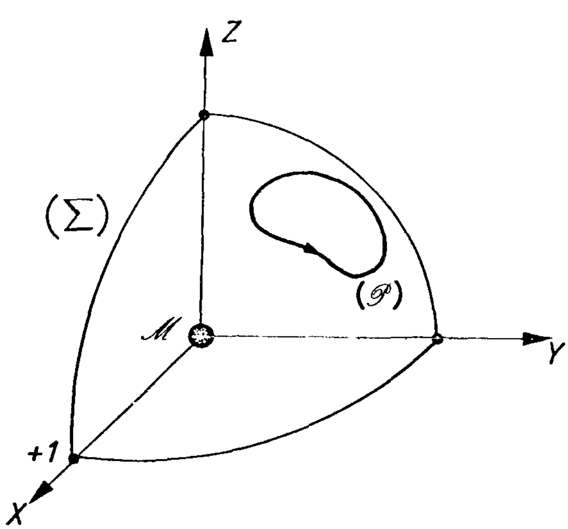


Fig. 10. Singular Case.

The Maximum Principle does not permit, *at least in the linearized study*, a determination in the singular case of the modulus F of the thrust between the values 0 and F_{max}, nor the position of the thrust arcs. Therefore it is necessary to have recourse to a direct study and the solution is not generally unique but *degenerates* into a large number of solutions. This "degeneracy" is generally removed by a higher order study (see Chapter II,4. and § II,5.3.). Therefore it is quite necessary to distinguish these singular cases of the *linearized problem* from the singular case pointed out in § I,2.3. leading to "intermediate thrust arcs".

In order for such a singular case to be presented, it is both necessary and sufficient for the polynomial P_6' (cos u) given in (61) to be identically zero, which leads to the conditions: /48

$$
\begin{aligned}
(a) &\equiv e\left(p_\xi^2 - \frac{p_\eta^2}{b^2} - b^2 p_\alpha^2 + p_\beta^2\right) = 0 \\
(b) &\equiv -p_\xi^2 + \frac{1+2e^2}{b^2} p_\eta^2 + 3b^2 p_\alpha^2 + \left(2e^2 - 3\right) p_\beta^2 = 0 \\
(c) &\equiv e - e p_\xi^2 - \frac{e\left(2+e^2\right)}{b^2} p_\eta^2 + 4e\, p_a^2 + 8b^2 p_a p_\alpha - 3eb^2 p_\alpha^2 - e\left(2-e^2\right) p_\beta^2 = 0 \\
(d) &\equiv -1 + p_\xi^2 + \frac{e^2}{b^2} p_\eta^2 + 4p_a^2 + b^2 p_\alpha^2 + \left(4 - 3e^2\right) p_\beta^2 = 0 \\
(f) &\equiv e\left(\frac{p_\xi p_\eta}{b} - b p_\alpha p_\beta\right) = 0 \\
(g) &\equiv -\frac{\left(1+e^2\right)}{b} p_\xi p_\eta + b\left(3-e^2\right) p_\alpha p_\beta = 0 \\
(h) &\equiv \frac{e}{b} p_\xi p_\eta + 4b\, p_a p_\beta - b e p_\alpha p_\beta = 0
\end{aligned}
\qquad (62)
$$

(with $b = \sqrt{1 - e^2}$)

These equations can only be verified simultaneously if $e = 0$ and:

$$p_a = \frac{\varepsilon_a}{2}\left(\varepsilon_a = \pm 1\right), \; p_e = p_z = 0 \qquad (63) \text{ I bis}$$

$$p_a = 0, \; p_e = \frac{1}{2}, \; p_z = \frac{\sqrt{3}}{2}, \; L_z = \begin{cases} L_e \\ L_e + \pi \end{cases} \qquad (64) \text{ III}$$

Therefore there is no singular solution (of the linearized problem) for $e \neq 0$. /49

In the circular case $e = 0$, there are two types of singular solutions:

The type I bis corresponds to an efficiency curve (P) reduced to a point ω situated at one of the intersection points ($y = \varepsilon_a = \pm 1$) of the sphere (Σ) and of the axis $\overleftrightarrow{MY}$ (Figure 11). Therefore the optimal thrust is tangential. This case will be studied in detail in § II,1.3.2.2., II,3.3.2.5. and II,5.2.1.3.

In *type III*, the efficiency curve (P) coincides with one of the large circles $(C^+)(L_z - L_e = 0)$ or $(C^-)(L_z - L_e = \pi)$ of (Σ) situated in the planes (π^+) or (π^-) tangent to the orbit and forming angles equal to $\pm 30°$ with the local horizontal plane X = 0 (Figure 12).

The optimal thrust is situated in one of these planes. At points $L = L_e$ and $L_e + \pi$ the thrust is tangential ($X = Z = 0$, $Y = \pm 1$).

This case will be studied in detail in § II,5.2.1.3.

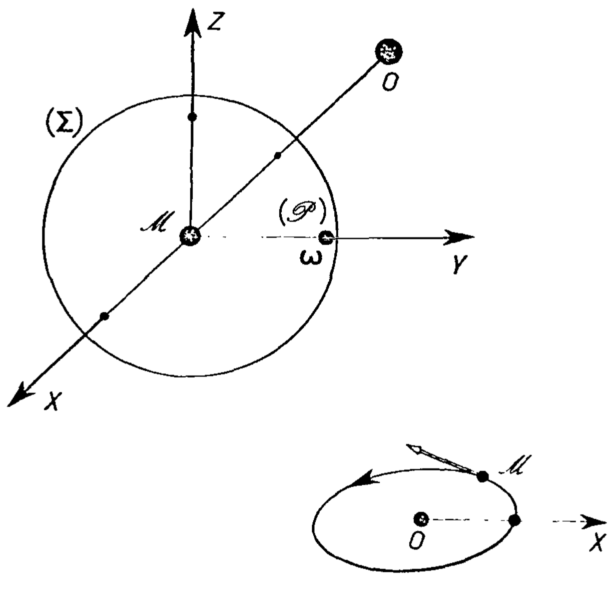


Fig. 11. Type I bis.

I,3.3. Determination of the Variations in the Orbital Elements as a Function of the Adjoint (Integration). /50

Bringing the law of optimal thrust which has just been determined as function of the adjoint (§ I,3.2.) into the equations of movement (36) we get in a matrix form:

$$\delta \mathcal{X}' = rK\left[\gamma\right] = r\frac{\gamma}{|\overrightarrow{Pv}|}Kp_V^T = \frac{\gamma}{|\overrightarrow{Pv}|}rKK^TP^T = \frac{\gamma}{|\overrightarrow{Pv}|}BP^T \qquad (65)$$

whence, by integrating u_0 to u_f:

$$\boxed{\Delta \mathcal{X} = GP^T} \qquad (66)$$

with:

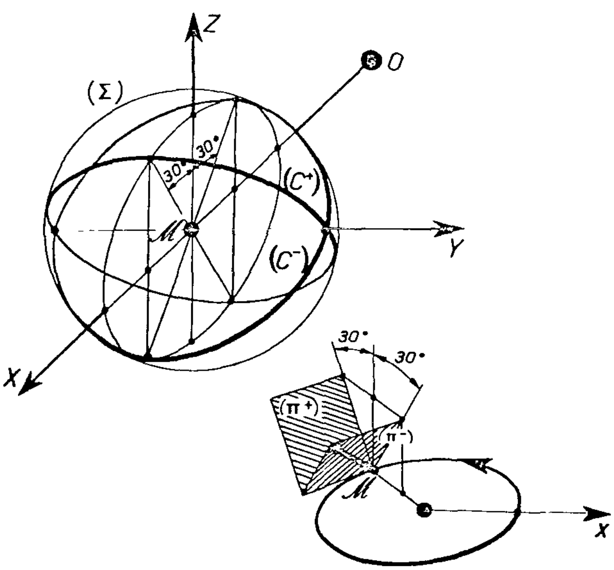


Fig. 12. Type III.

$$G = \int_{u_0}^{u_f} \frac{\gamma}{|\vec{P_V}|} B\,du = \begin{bmatrix} G_{\xi\xi} & G_{\xi\eta} & 0 & 0 & 0 & 0 \\ G_{\eta\xi} & G_{\eta\eta} & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{aa} & G_{a\alpha} & G_{a\beta} & G_{a\tau} \\ 0 & 0 & G_{\alpha a} & G_{\alpha\alpha} & G_{\alpha\beta} & G_{\alpha\tau} \\ 0 & 0 & G_{\beta a} & G_{\beta\alpha} & G_{\beta\beta} & G_{\beta\tau} \\ 0 & 0 & G_{\tau a} & G_{\tau\alpha} & G_{\tau\beta} & G_{\tau\tau} \end{bmatrix} \quad \text{square symmetrical matrix } (6 \times 6) \qquad (67)$$

The equation giving the consumption is written then:

$$\boxed{\Delta C = \int_{u_0}^{u_f} \gamma_{max}\, U(\Theta)\, r\, du} \qquad \left(\Theta = |\vec{P_V}| - 1\right) \qquad (68)$$

or else:

$$\Delta J = \frac{1}{2}\int_{u_0}^{u_f} r\, \vec{P_V}^2\, du = \frac{1}{2}\int_{u_0}^{u_f} r\, PKK^T P^T = \frac{1}{2}\int_{u_0}^{u_f} PBP^T du$$

or

$$\boxed{\Delta J = \frac{1}{2} PGP^T} \qquad (69)$$

1,3.3.1. Systems (S_2).

Calculation of the matrix G is simple for the systems (S_2). As a matter /51
of fact in this case $\gamma = |\vec{p}_V|$ and it suffices to integrate matrix B of which the elements are polynomials in sin u, cos u and u, which presents no difficulty except the length of the calculations.

The elements of G are given in Appendix 8.

1,3.3.2. Systems (S_1)

On the other hand, in the case of the systems (S_1), $\gamma = \gamma_{max}\, U\,(\Theta)$ and $|\vec{p}_V|$ continues to exist in the denominator of the integrand. Therefore it is first necessary to determine the commutation points (roots of $\Theta = 0$), and then to calculate an integral in complex form.

The integration can only be carried out in a certain number of particular cases.

I,3.4. Search for the Adjoint Beginning With Variations in the Orbital Elements (Inversion).

The resolution of the optimization problem implies finally the a posteriori determination of the non-zero components, thus unknown ones, of the kinematic adjoint (and of the non-imposed variations, thus unknown ones, of the orbital elements) as a function of the imposed variations of the orbital elements. This allows the law of optimal thrust (directrix curve, efficiency curve) to be set and, on the other hand, consumption to be calculated.

I,3.4.1. Systems (S_2).

In the case of systems (S_2), this determination is easy because the *matrix* G *does not contain the kinematic adjoint* P.

System (66) is linear in respect to the unknowns (non-zero components of the kinematic adjoint and non-imposed variations of the orbital elements).

Therefore the problem of optimization is completely solvable in the general case for propulsion systems (S_2) as Ross and Leitmann [36] have pointed out in the case of transfer and as Edelbaum [37] has pointed out in the case of rendezvous.

For example, if all the variations of the orbital elements are imposed, the kinematic adjoint P is obtained by inversion of the matrix equation (66):

$$P = \Delta \mathscr{X}^T G^{-1} \tag{70}$$

Introducing this value into the consumption equation, we get

$$\boxed{\Delta J = P_{max} \, |\Delta m| = \frac{1}{2} \Delta \mathscr{X}^T G^{-1} \Delta \mathscr{X}} \tag{71}$$

The performance index ΔJ is therefore a *quadratic form of the variations of the orbital elements.*

Presentation of the results can be envisaged in the following manner, positing:

$$\zeta = \frac{\Delta \mathscr{X}}{\sqrt{2 P_{max} \, |\Delta m| \Delta t}} \tag{72}$$

equation (71) becomes:

$$\boxed{\zeta^T <G>^{-1} \zeta = 1} \qquad (73)$$

where $<G> = \dfrac{G}{\Delta t}$ and $\Delta t = u_f - e \sin u_f - (u_o - e \sin u_o)$.

This equation is a hyperquadratic one in the six-dimensional space of the "*reduced variations*" ζ.

The general discussion of the problem is reduced to the study of the de- /52
formations of this hyperquadratic when Δu varies, for a value of the eccentricity e and a fixed initial position u_o (altogether three parameters for discussion).

In the case of a whole number N of revolutions, the discussion only brings in the parameter e in the case of transfers [see equation (I,4-3), but the three parameters e, u_o and Δu in the case of rendezvous.

For a large number of revolutions [N $\geq$ order $\frac{1}{\varepsilon}$] the discussion only brings in the parameter e in both the case of transfers and in that of rendezvous [if $\Delta\tau_1/\Delta u$ is taken as a sixth component; see equation (I,4-10)].

In the case where the nominal orbit is circular (e = 0), the discussion now only brings in ΔL (see § II,3.3.1.) (take axis $\vec{Ox}$ following the axis of symmetry of the transfer arc $\widehat{M_o M_f}$).

If in addition there is a complete number N of revolutions, no discussion is necessary in the case of transfers and there is a discussion as a function of ΔL (or N) in the case of rendezvous.

Finally, in the case of a large number of revolutions [N $\geq$ order $\frac{1}{\varepsilon}$], no discussion is necessary (if $\Delta\tau_1/\Delta L$ is taken as a sixth component).

Let us note that, by using the reduced variations ζ, the discussion does not bring in any parameter relating to the propulsion system.

When certain variations of the orbital elements are not imposed, it is enough to be concerned with deformations when Δu varies (for determined u_o and e) of the *projection* of the hyperquadratic defined above on the hyperspace (or space, plane, straight line) of the determined variations, parallel to the hyperspace (or space, plane, straight line) of the indifferent variations.

I,3.4.2. Systems (S_1).

In the case of the systems (S_1), the determination envisaged at the beginning of § I,3.4. is delicate because the matrix G does not only depend on e, u_o and u_f *but also on the kinematic adjoint* P; in general it cannot be carried out except by a calculation of *successive approximations*.

However, if this determination is possible, we get an analytical expression of the kinematic adjoint P in the form:

$$P = P\left(e, u_o, u_f = u_o + \Delta u, \lambda = \frac{\Delta X}{F_{max}\,\Delta t}\right) \tag{74}$$

which a posteriori determines the optimal thrust law (here ΔX only represents the *imposed* part of the variations of the elements). Introducing this value into the consumption equation, we get:

$$\lambda_c = \frac{\Delta C}{F_{max}\,\Delta t} = \frac{W/\Delta m/}{F_{max}\,\Delta t} = \lambda_c\left(e, u_o, u_f = u_o + \Delta u, \lambda\right) \tag{75}$$

[For a complete number of revolutions $\Delta u = 2N\pi$ in the elliptical case ($e \neq 0$) or for any angle of transfer in the circular case ($e = 0$), Δt coincides with Δu; λ_c is nothing but the relationship $\Delta t_{prop}/\Delta t$ of the total duration of the maximal thrust arcs on the duration of the transfer].

Thus we obtain an explicit formula giving consumption as a function of the transfer parameters: e, u_o, u_f, ΔX (imposed) and the maximal thrust F_{max} which only occurs in the relationships λ.

For this reason it is useful to introduce "*specific variations*":

$$\nu = \frac{\Delta X}{\Delta C} = \frac{\Delta X}{W/\Delta m/} = \frac{\lambda}{\lambda_c} = \nu\left(e, u_o, u_f = u_o + \Delta u, \lambda\right) \tag{76}$$

i.e. variations per unit of characteristic velocity. /53

It is likewise useful to consider the reduced variations ζ [introduced previously in the case of systems (S_2)] in order to be able to compare performance of the two propulsion types.

Let us note that ν and ζ are connected by:

$$\zeta = \nu\sqrt{\lambda_c} \tag{77}$$

and that $\zeta = \nu$ for $\lambda_c = 1$, i.e. when the maximal thrust F_{max} is applied in a continuous fashion.

NOTE:

The parameter ν is of more interest than the parameter ζ for impulsional solutions, because then $\nu \to$ of a finite limit $\neq 0$, while $\zeta \to 0$.

The complete discussion of the problem brings not only the parameters e, u_o and Δu [which intervened in the discussion for the system (S_2)], but also ali of the imposed parameters λ.

The same sort of simplifications as in the case of system (S_2) is introduced by the hypotheses: whole number or large number of revolutions, eccentricity of the nominal orbit equal to 0, etc...

If analytical inversion is impossible, it is necessary to have recourse to calculation by *successive approximations*, possibly facilitated by tracing ahead of time a certain number of nomographic charts in the adjoint space P. If the number of non-zero components of P is equal to 2, we trace the lines:

$$\lambda = \frac{\Delta \mathcal{X}}{F_{max}\Delta t} = \lambda\left(e, u_o, \Delta u, P\right) = C^{te} \tag{78}$$

and

$$\lambda_c = \frac{\Delta C}{F_{max}\Delta t} = \lambda_c\left(e, u_o, \Delta u, P\right) = C^{te} . \tag{79}$$

Thus knowing e, u_o, Δu, ΔX imposed, F_{max}, we deduce from them the values of λ, whence P (and optimal thrust) and λ_c (consumption) by interpolation. An example of such a solution is given in § II,2.3.2.2.

I,3.5. Criterion for Comparing Performance of Propulsion System (S_1) and (S_2).

In order to show the penalty brought about by not modulating the ejection velocity, a propulsion system (S_2) can be compared for the mission given to system (S_1) *of the same installed power* P_{max} and with constant ejection velocity W. In order not to multiply the number of systems (S_1) serving for comparision, we shall only retain two types of them:

1. *System* (S_{1c}): the ejection velocity W is such that the maximal thrust F_{max} is applied in a continuous fashion.
2. *System* $(S_1{}^*)$: ejection velocity W is equal to the optimal ejection velocity W* for the mission considered, given by the equation (I,2-43) which is written in the linearized study:

$$\int_{t_0}^{t_f} U(\Theta)\left(|\vec{P_V}|-2\right)dt = \int_{u_0}^{u_f} U(\Theta)\left(|\vec{P_V}|-2\right)r\,du = 0\,. \tag{80}$$

These are the reduced variations $\zeta = \dfrac{\Delta X}{\sqrt{2P_{max}|\Delta m|\Delta t}}$ which will serve /54 as a base of comparison between the systems (S_1) and (S_2), since their definition involves as a propulsion parameter only the power P_{max}, which by hypothesis is the same for all these systems.

Let us note that for the systems (S_1) of the same power P_{max},

$$|\zeta| = \sqrt{\lambda\nu} = f\left(e, u_0, u_f = u_0 + \Delta u, \Delta X, W\right)$$

and since the transfer (e, u_o, $u_f = u_o + \Delta u$, ΔX) is given, the parameters $|\zeta|$ are only functions of the ejection velocity and are simultaneously the maximums for $W = W^*$, which furnishes a means for calculating W^* without using equation (80) (see § II,1.3.1.3. and II,2.3.2.3.).

In concluding this chapter let us emphasize the complexity of the problem of transfer between close orbits in the case of a propulsion system (S_1) with a constant ejection velocity compared to the case of a propulsion system (S_2) with a modulable ejection velocity.

While in the last case the problem is solvable for a transfer with any rendezvous whatsoever, the solution in the first case comes up against three essential difficulties:

- determination of the commutation points (roots of the equation $\Theta = 0$ of degree ≤ 6),
- complicated integration,
- difficult inversion.

Thus there can be no hope of resolving it except in a certain number of particular cases (Second Part). Nevertheless a few general results concerning uncoupling can be stated. They are the object of the following chapter.

I,4. GENERAL RESULTS. UNCOUPLING

We separate the study referring to propulsion systems (S_2) where uncoupling is frequent from that referring to propulsion systems (S_1) where only "non-induction" can occur.

I,4.1. Case of Propulsion Systems (S_2) (Modulable Ejection Velocity).

I,4.1.1. Uncoupling between the rotation of the plane of the orbit and modifications in the plane of the orbit.

The shape (I,3-67) of the matrix G, which can be decomposed into two square and symmetrical matrices 2 × 2 and 4 × 4, shows that when no modification is imposed in the plane of the orbit ($p_a = p_\alpha = p_\beta = p_\tau = 0$), no such modification appears to be *induced* by the rotation $\Delta\xi$, $\Delta\eta$ of the plane of the orbit.

Inversely, if the rotation of the plane of the orbit is not imposed ($p_\xi = p_\eta = 0$), no *induced* rotation of this plane appears through modifications Δa, $\Delta\alpha$, $\Delta\beta$, $\Delta\tau$ in the plane of the orbit.

Since the matrix G does not depend on the adjoint P, there is also /55 *uncoupling* between the problem of the rotation of the plane of the orbit and the problem of the modifications of the orbit in its plane: the values of the components P_ξ, P_η, P_a, P_α, P_β, P_τ of P calculated from the variations $\Delta\xi$, $\Delta\eta$, Δa, $\Delta\alpha$, $\Delta\beta$, $\Delta\tau$ are nothing but the values P_ξ, P_η on the one hand and Pa, $P\alpha$, $P\beta$, $P\tau$ on the other which would be obtained from the variations $\Delta\xi$, $\Delta\eta$ on the one hand and Δa, $\Delta\alpha$, $\Delta\beta$, $\Delta\tau$ on the other, by *successively* resolving the problem of the rotation of the plane and the problem of the modifications of the orbit in its plane.

Equation (I,3 -41) shows that the optimal acceleration relative to the total problem is the vectorial sum of the optimal accelerations relative to the component problems. Likewise, equation (I,3 -69) shows that amounts of consumption are added algebraically.

I,4.1.2. Case of a complete number of turns.

The elements of the matrix G given in Appendix 8 show that in the case where $\Delta u = u_f - u_0 = 2N\pi$, the minor relative to the element $G_{\tau\tau}$ of G is diagonal.

$$G = \begin{bmatrix} G_{\xi\xi} & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{\eta\eta} & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{aa} & 0 & 0 & G_{a\tau} \\ 0 & 0 & 0 & G_{\alpha\alpha} & 0 & G_{\alpha\tau} \\ 0 & 0 & 0 & 0 & G_{\beta\beta} & G_{\beta\tau} \\ 0 & 0 & G_{\tau a} & G_{\tau\alpha} & G_{\tau\beta} & G_{\tau\tau} \end{bmatrix} \quad (1)$$

In the case of a transfer without rendezvous ($p_\tau = 0$), if the variation of one of the first five orbital elements ξ, η, a, α, β is imposed, there is no variation induced in the four other elements. There is even uncoupling between the variations of the first five elements in the sense that we have given this word in the previous paragraph.

As $p_\tau = 0$, equation (I,3 - 69) shows that only the minor relative to the elements $G_{\tau\tau}$ of G intervenes in calculating the consumption ΔJ. Since this minor is diagonal, ΔJ is a *linear and homogeneous function of the squares of the variations* $\Delta\xi$, $\Delta\eta$, Δa, $\Delta\alpha$, $\Delta\beta$.

$$\Delta J = \frac{1}{2}\left(\frac{\Delta\xi^2}{G_{\xi\xi}} + \frac{\Delta\eta^2}{G_{\eta\eta}} + \frac{\Delta a^2}{G_{aa}} + \frac{\Delta\alpha^2}{G_{\alpha\alpha}} + \frac{\Delta\beta^2}{G_{\beta\beta}}\right) =$$
$$\frac{1}{2\Delta u}\left[2\Delta\xi^2 + \frac{2(1-e^2)}{1+4e^2}\Delta\eta^2 + \frac{\Delta a^2}{4} + \frac{2\Delta\alpha^2}{5(1-e^2)} + \frac{2\Delta\beta^2}{5-4e^2}\right]. \qquad (2)$$

The hyperquadratic envisaged in § I,3.4.1.has as an equation (in the space of five reduced variations ζ)

$$2\zeta_\xi^2 + \frac{2(1-e^2)}{1+4e^2}\zeta_\eta^2 + \frac{\zeta_a^2}{4} + \frac{2\zeta_\alpha^2}{5(1-e^2)} + \frac{2\zeta_\beta^2}{5-4e^2} = 1. \qquad (3)$$

In the case $p_\tau \neq 0$ (which can only be produced in the case of rendezvous) /56 equation (I,3 - 69) shows that ΔJ does not have such a simple shape because G is not diagonal.

However it is demonstrated in Appendix 9 that if

$$\Delta\tau_1 = \Delta\tau - \Delta\tau_{t.s.} \qquad (4)$$

is chosen as the sixth variation, and no longer $\Delta\tau$, referring to rendezvous, the new matrix G_1 is diagonal.

In (4), $\Delta\tau_{t.s.}$ represents the variation of τ in the optimal *simple transfer* corresponding to the rendezvous under consideration, induced by the variations of a, α and β.

$\Delta\tau_1$ is the necessary *supplement* to assure rendezvous.

If the new variation in the kinematic state is:

$$\Delta \mathcal{X}_1 = \begin{vmatrix} \Delta\xi \\ \Delta\eta \\ \Delta a \\ \Delta\alpha \\ \Delta\beta \\ \Delta\tau_1 \end{vmatrix} \tag{5}$$

then:

$$\Delta J = \frac{1}{2} \left(\Delta \mathcal{X}_1\right)^T G_1^{-1} \Delta \mathcal{X}_1 \tag{6}$$

where G_1 is, this time, a diagonal matrix. Whence:

$$\Delta J = \Delta J_{t.s.} + \Delta J_1 = \Delta J_{t.s.} + \frac{\Delta\tau_1^2}{2 G_{1\tau\tau}} \tag{7}$$

where $\Delta J_{t.s.}$ is given in (2) and where:

$$G_{1\tau\tau} = G_{\tau\tau} - \frac{G_{a\tau}^2}{G_{aa}} - \frac{G_{\alpha\tau}^2}{G_{\alpha\alpha}} - \frac{G_{\beta\tau}^2}{G_{\beta\beta}} \underset{\Delta u \to \infty}{\sim} \frac{3}{4} \Delta u^3. \tag{8}$$

Therefore, for a large number of turns:

$$\Delta J_1 \sim \frac{2\Delta\tau_1^2}{3\Delta u^3}. \tag{9}$$

The hyperquadratic envisaged in § I,3.4.1. has as an equation (in the space of the six reduced variations ζ):

$$2\zeta_\xi^2 + \frac{2(1-e^2)}{1+4e^2}\zeta_\eta^2 + \frac{\zeta_a^2}{4} + \frac{2\zeta_\alpha^2}{5(1-e^2)} + \frac{2\zeta_\beta^2}{5-4e^2} + \frac{4\zeta_1^2}{3\Delta u^2} = 1. \tag{10}$$

Again we find the fact that it not $\Delta\tau_1$ but rather $\Delta\tau_1/\Delta u$ which is to be compared with the variations of the first five elements. As $\Delta\tau_1 < \pi$, when $N \gg \frac{1}{\varepsilon}$, the rendezvous only costs a negligible supplement.

I,4.1.3. Case of any transfer angle.

It would evidently be desirable to extend the preceding results concerning uncoupling to cases where the mobile does not carry out a whole number of turns (nor a large number of turns).

Let us remark immediately that complete uncoupling between the rotation of the plane of the orbit and the modifications in the plane of the orbit permits a choice of axes where $\vec{\Delta Z}_{plane}$ and $\vec{p}_Z$ are adjusted independently of those where $\vec{\Delta e}_{plane}$ and $\vec{p}_e$ are adjusted.

If, in order to adjust $\vec{\Delta Z}_{plane}$ and $\vec{p}_Z$, we no longer take the axes $\vec{Ox}$, $\vec{Oy}$ as previously but the axes $\vec{Ox}_1$, $\vec{Oy}_1$ (from the plane $\vec{Ox}$, $\vec{Oy}$) which cut across the symmetrical matrix 2 × 2 in a diagonal manner: /57

$$G_1 = \begin{vmatrix} G_{\xi\xi} & G_{\xi\eta} \\ G_{\eta\xi} & G_{\eta\eta} \end{vmatrix} \tag{11}$$

the variations $\Delta\xi_1$ and $\Delta\eta_1$ are uncoupled.

Likewise, in choosing the axes $\vec{Ox}_2$, $\vec{Oy}_2$ (of the plane Ox, Oy) which cut across the symmetrical matrix 2 × 2 in a diagonal manner, in order to adjust $\vec{\Delta e}_{plane}$ and $\vec{p}_e$:

$$G_2 = \begin{vmatrix} G_{\alpha\alpha} & G_{\alpha\beta} \\ G_{\beta\alpha} & G_{\beta\beta} \end{vmatrix} \tag{12}$$

we uncouple the variations $\Delta\alpha_2$ and $\Delta\beta_2$. This last choice does not have a great deal of interest for $e \neq 0$, because Δa and $\Delta\alpha_2$ on the one hand and Δa and $\Delta\beta_2$ on the other remain coupled and moreover, $\Delta\alpha_2$ and $\Delta\beta_2$ can no longer be simply reconnected, as $\Delta\alpha$ and $\Delta\beta/e$ were previously, to the eccentricity variation Δe and to the rotation of the orbit in its plane.

In the case where the nominal orbit is circular ($e = 0$), the axes $\vec{Ox}_1$ and $\vec{Ox}_2$ coincide with the axis of symmetry of the transfer arc $\widehat{M_0M_f}$ which is then evidently of interest to be taken as reference axis $\vec{Ox}$.

1,4.2. Case of Propulsion Systems (S_1) (Constant Ejection Velocity)

1,4.2.1. Mutual non-induction between the rotation of the plane of the orbit and modification of the plane of the orbit.

The shape of the matrix G (which can be broken down into two square symmetrical matrices 2 × 2 and 4 × 4) resembles that which refers to systems (S_2). When no modification is imposed in the plane of the orbit ($p_a = p_\alpha = p_\beta = p_\tau = 0$), there does not appear any such *induced* modification by the rotation $\Delta\xi$, $\Delta\eta$ of the plane of the orbit. Inversely, if the rotation of the plane of the orbit is not imposed ($p_\xi = p_\eta = 0$) no rotation of this plane appears *induced* by the modifications Δa, $\Delta\alpha$, $\Delta\beta$, $\Delta\tau$ in the plane of the orbit.

However, contrary to the result obtained for the system (S_2), there is *no uncoupling* between the rotation of the plane of the orbit and the modifications in the plane of the orbit. As a matter of fact *the matrix* G *depends on the adjoint* P which is not the same, depending on whether the problem of the rotation, the problem of the modifications in the plane or the problem of total transfer is considered. These solutions are not additive.

I,4.2.2. Case of a Whole Number of Turns.

The property of *non-induction* depends on the zero elements of the matrix G. In addition to the zero elements occurring in (I,3 - 67) and explaining the results obtained in the previous paragraph, other elements can nullify themselves under particular conditions.

For example, let us consider the case of a whole number N of turns ($\Delta u = 2N\pi$) for transfers such that $e \neq 0$. If $|\vec{\vec{p}}_V|$ is a function of cos u alone (in Appendix 7 this case corresponds to $Q \equiv 0$), the elements of the minor of G referring to element $G_{\tau\tau}$ corresponding to the elements of B of the shape sin u Q (cos u) are zero. As a matter of fact such an element is written:

$$G_{ij} = \int_{u_0}^{u_f = u_0 + 2N\pi} \gamma_{max} \frac{U(|\vec{p_V}| - 1)}{|\vec{p_V}|} Q(\cos u) \sin u \, du = \int_{u_0}^{u_0 + 2N\pi} F(\cos u) \sin u \, du = 0 \quad (13)$$ /58

Under these conditions the minor of $G_{\tau\tau}$ is written:

$$\text{minor of } G_{\tau\tau} = \begin{array}{|c|c|c|c|c|} \hline G_{\xi\xi} & 0 & 0 & 0 & 0 \\ \hline 0 & G_{\eta\eta} & 0 & 0 & 0 \\ \hline 0 & 0 & G_{aa} & G_{a\alpha} & 0 \\ \hline 0 & 0 & G_{\alpha a} & G_{\alpha\alpha} & 0 \\ \hline 0 & 0 & 0 & 0 & G_{\beta\beta} \\ \hline \end{array} \quad (14)$$

It is important to notice that the element $G_{a\alpha}$ (= $G_{\alpha a}$) is not zero and that (for $e \neq 0$) there is no mutual induction between the variations Δa and $\Delta\alpha$.

For each of the thirty-one classes of transfer (without rendezvous) corresponding to a total of imposed variations of the first five orbital elements (the possible induced variations of the other elements being considered indifferent), Appendix 7 indicates which transfer classes definitely present no induced variations and what the possible induced variations concerning other classes of transfer are.

For the following eleven classes of transfers, there is no other variation than those which are imposed. It is not certain that these are the only ones, but it appears probable:

$\Delta\xi$ (rotation around the "parameter" $\overrightarrow{Oy}$),
$\Delta\eta$ (rotation around the major axis $\overrightarrow{Ox}$),
$\Delta\beta$ (rotation of the orbit in its plane),
$\Delta\xi$, $\Delta\eta$ (rotation of the plane of the orbit),
$\Delta\xi$ $\Delta\beta$ (rotation around an axis contained in the plane Oyz),
$\Delta\eta$, $\Delta\beta$ (rotation around an axis contained in the plane Oxz),
Δa, $\Delta\alpha$ (plane coaxial transfers),
$\Delta\xi$, Δa, $\Delta\alpha$ (co-parameter transfers),
$\Delta\eta$, Δa, $\Delta\alpha$ (coaxial transfers),
Δa, $\Delta\alpha$, $\Delta\beta$ (plane transfers),

and obviously:

$\Delta\xi$, $\Delta\eta$, Δa, $\Delta\alpha$, $\Delta\beta$ (general transfer).

In the case of transfers between near-circular orbits ($e \leq$ order $\varepsilon \ll 1$), where the nominal orbit is circular ($e = 0$), non-induction is more common.

There are only 19 cases of transfer to consider instead of 31, because the choice of the axis of reference is free.

For example, the transfer class where only $\Delta\xi$ is imposed corresponds, to except for one rotation, to the transfer class where $\Delta\eta$ is only imposed.

On the other hand, a transfer class where $\Delta\xi$ and $\Delta\eta$ are imposed simultaneously is obviously not reduced to one of the above classes. Taking /59
$\overrightarrow{Ox}$ according to $\overrightarrow{\Delta Z}_{plane}$, it corresponds to $\Delta\xi$ imposed and $\Delta\eta = 0$, therefore likewise imposed, a condition which does not figure in the above.

In Appendix 7, classes of double utilization have been mentioned.

1. If $|\overrightarrow{p}_V|$ is uniquely a function of $\sin^2 L$ or $\cos^2 L$, i.e. finally a function of sin L or cos L alone, the elements of the minor of G referring to elements $G_{\tau\tau}$ corresponding to elements of B of the shape sin L Q (cos L) or cos L R (sin L) (where Q and R are polynomials) are zero. Then the minor is

diagonal:

$$\text{minor of } G_{\tau\tau} = \begin{array}{|c|c|c|c|c|}\hline G_{\xi\xi} & 0 & 0 & 0 & 0 \\ \hline 0 & G_{\eta\eta} & 0 & 0 & 0 \\ \hline 0 & 0 & G_{aa} & 0 & 0 \\ \hline 0 & 0 & 0 & G_{\alpha\alpha} & 0 \\ \hline 0 & 0 & 0 & 0 & G_{\beta\beta} \\ \hline \end{array} \qquad (15)$$

The variations imposed do not induce any other variation.

2. If $|\vec{\vec{p}}_V|$ is a function of sin L alone, the elements of the minor of G referring to element $G_{\tau\tau}$ corresponding to elements of B of the shape cos L R (sin L) are zero.

Then the minor is written:

$$\text{minor of } G_{\tau\tau} = \begin{array}{|c|c|c|c|c|}\hline G_{\xi\xi} & 0 & 0 & 0 & 0 \\ \hline 0 & G_{\eta\eta} & 0 & 0 & 0 \\ \hline 0 & 0 & G_{aa} & 0 & G_{a\beta} \\ \hline 0 & 0 & 0 & G_{\alpha\alpha} & 0 \\ \hline 0 & 0 & G_{\beta a} & 0 & G_{\beta\beta} \\ \hline \end{array} \qquad (16)$$

3. If $|\vec{\vec{p}}_V|$ is a function of cos L alone, the elements of the minor of G referring to element $G_{\tau\tau}$ corresponding to elements of B of the shape sin L Q (cos L) are zero. Then the minor is written:

$$\text{minor of } G_{\tau\tau} = \begin{array}{|c|c|c|c|c|}\hline G_{\xi\xi} & 0 & 0 & 0 & 0 \\ \hline 0 & G_{\eta\eta} & 0 & 0 & 0 \\ \hline 0 & 0 & G_{aa} & G_{a\alpha} & 0 \\ \hline 0 & 0 & G_{\alpha a} & G_{\alpha\alpha} & 0 \\ \hline 0 & 0 & 0 & 0 & G_{\beta\beta} \\ \hline \end{array} \qquad (17)$$

4. Finally if $|\vec{p}_V|$ is a function of sin L and cos L, G has the form (I,3 - 67). /60

The properties of non-induction figuring in Appendix 7 are easily deduced from these.

Out of the 19 transfer classes to be considered, there are at least 14 for which the variations imposed do not induce any other variation.

This is particularly true for all the classes where the variation of a single element or even variations of two elements are imposed.

This last result is evident for the classes envisaged, except perhaps for the class where $\Delta\alpha$ and $\Delta\beta$ are imposed simultaneously or the class where $\Delta\xi$, $\Delta\eta$ and Δa are imposed at the same time. As a matter of fact $|\vec{p}_V|$ is then a function of sin L and cos L and, for example for the first class mentioned above, it is not a priori evident that there is no induced variation Δa. In fact, having found that in the transfers where only $\Delta\alpha$ is imposed there is no variation Δa and $\Delta\beta$ induced by this variation, there is, equivalence, except for, one rotation, between the problem where $\Delta\alpha$ alone is imposed and the problem where $\Delta\alpha$ and $\Delta\beta = 0$ are imposed (although these two problems do not form part of the same class). Since for the first there is no induced variation Δa, there is likewise none for the second.

The same reasoning holds for the second class ($\Delta\xi$, $\Delta\eta$, Δa) mentioned.

I,4.2.3. Case of any transfer angle between near-circles.

In the case e = 0, the choice of the axis of reference $\vec{Ox}$ is arbitrary. It is practical to take $\vec{Ox}$ according to the axis of symmetry of the transfer arc $\widehat{M_0M_f}$. If then the problem to be treated is, *with this choice of axis*, such that $|\vec{p}_V|$ can be considered as a function of cos L alone, the elements of the minor of G relating to element $G_{\tau\tau}$ corresponding to elements of B of the form sin L Q (cos L)(Q = polynomial) are zero. Then the minor has the shape (17).

This property is particularly true when the rotation $\Delta\eta$ around $\vec{Ox}$ (or $\Delta\xi$ around $\vec{Oy}$) is imposed alone. Then there is no rotation induced around $\vec{Oy}$ or $\vec{Ox}$), nor obviously any induced variation of the other elements (see § II,2.3.1.2.).

II - PARTICULAR PROBLEMS

The difficulties met in the First Part of the study, at the time of the attempt to find an analytical solution of the general problem of optimal transfers between close orbits for propulsion systems with a constant ejection velocity, clearly show that the complete solution has no chance of being found except for particular classes of transfers. /61

The particular problems considered in this Second Part of the study refer to transfer classes which offer an evident practical interest.

Their complexity keeps increasing, because the number of orbital elements with imposed variation increases and goes from one element (optimal infinitesimal variation of the semi-major axis; Chapter II, 1.) to two elements (optimal infinitesimal rotations of the plane of the orbit; Chapter II,2.), then to three elements (optimal transfers between close, coplanar, circular orbits; Chapter II,3.), to five elements (reduced to four by the hypothesis $e \approx 0$ *in the optimal impulsional transfers between close, near-circular orbits, whether coplanar or not; Chapter II,5.), and finally to six elements (reduced to five by the hypothesis* $e \approx 0$*, in long term rendezvous associated with transfers which have just been mentioned; Chapter II,6.).*

On the other hand, more and more significant simplifying hypotheses are made, of the type already mentioned in Chapter I,4.:

1. *whole number of revolutions,*
2. *large number of revolutions* ($N \geq$ order $\frac{1}{\varepsilon}$),
3. *elliptical orbits of low eccentricity* ($\varepsilon << e << 1$),
4. *near-circular orbits* ($e \leq$ order $\varepsilon << 1$),
5. *impulsional or quasi-impulsional solutions.*

In addition certain of these hypotheses can be cumulative.

Very fortunately hypotheses 2, 3, 4 and 5 correspond to cases which are found very often in practice (hypothesis 2 essentially for satellites).

The 3-step format of each of the first three studies and of certain parts of the two last ones is the same adopted in the general study of Chapter I,3.: determination of the optimal thrust as a function of the adjoint; determination of the variations of the orbital elements as a function of the adjoint (integration); search for the adjoint beginning with variations of the orbital elements (inversion).

In the first three studies the performance of propulsion systems (S_1) *and* (S_2) *are compared. In the first study (optimal variation of the semi-major axis) even the most general propulsion system* (S) *is envisaged in view of the relative simplicity of the problem.*

Chapter II,4. shows separately how the optimal solution can be determined by a higher order study when the linearized solution is singular.

II,1. OPTIMAL INFINITESIMAL VARIATION OF THE SEMI-MAJOR AXIS

II,1.1. Introduction.

This is a matter of achieving the variation Δa of the semi-major axis of the orbit, the variations of the other elements being considered as indifferent. The only component of the kinematic adjoint P which is not necessarily zero is p_a. This problem is important in practice (variation of the period of the orbit). /62

II,1.2. Optimal Thrust.

Equation (I,3 - 41) shows that the efficiency vector

$$\overrightarrow{P_V} = 2 p_a \vec{V} \tag{1}$$

is proportional to the velocity $\vec{V}$.

The acceleration of optimal thrust $\vec{\gamma}$ is therefore *borne by the tangent to the orbit* and directed forward if $p_a > 0$, backward if $p_a < 0$. It is anti-symmetrical in relation to the major axis of the orbit.

The efficiency curve (*P*) is a *circle* of the plane MXY (Figure 1a).

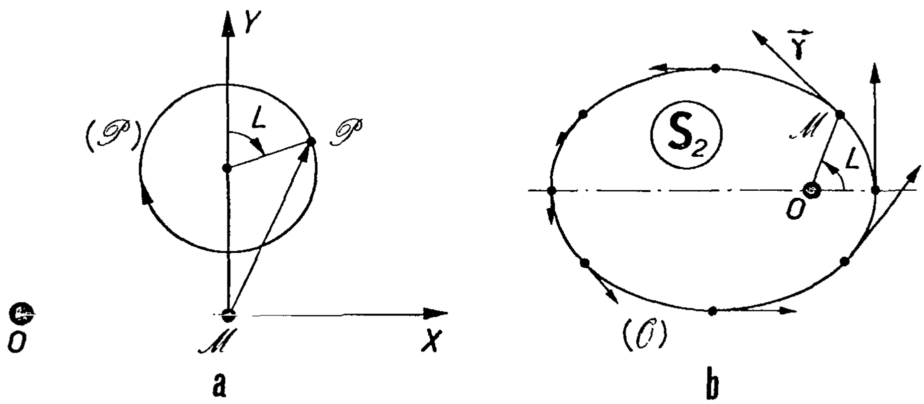


Fig. 1. a) Efficiency Curve; b) Optimal Acceleration (system S_2).

$$\overrightarrow{\mathcal{M}\mathcal{P}} = \overrightarrow{P_V} \begin{cases} X = \frac{2 p_a}{b} e \sin L \\ y = \frac{2 p_a}{b} (1 + e \cos L) \end{cases} \qquad (b = \sqrt{1 - e^2}). \tag{2}$$

It is known, in fact, that the hodograph of an elliptical movement is a circle when the turning axes are chosen as axes of reference (just as when the fixed axes are chosen).

The directrix orbit coincides with the orbit itself (see § I,3.2.2.).

For the propulsion system (S_2) optimal thrust acceleration $\vec{\gamma}$ is modulated proportionally to the velocity $\vec{V}$ (Figure 1b).

In the case of propulsion system (S_1), maximal thrust F_{max} is applied (if $\Delta L = 2N\pi$) on an arc symmetrical in relation to the perigee (Figure 2b). /63

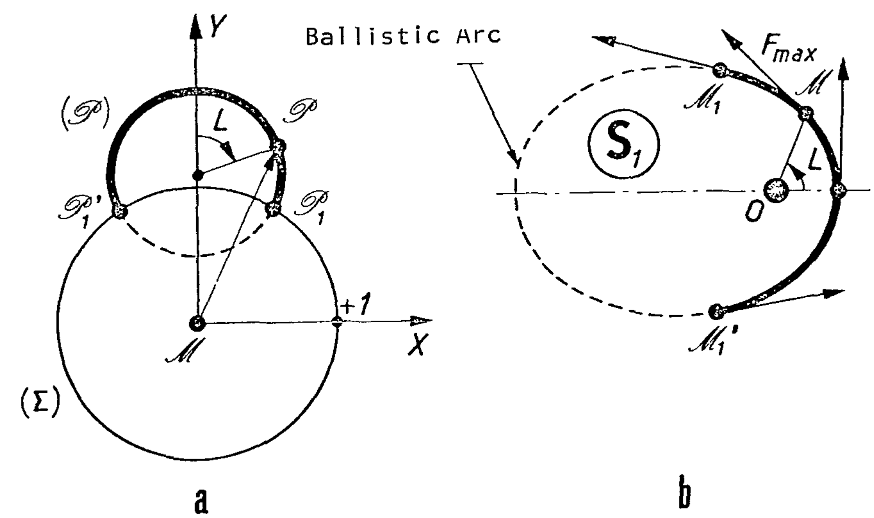


Fig. 2. a) Efficiency Curve; b) Optimal Thrust (Systems S_1).

Finally, in the most complicated case of propulsion system (S), a zone, for example, of maximal thrust F_{max} to the perigee can be found surrounded by two zones with modulated thrust, then by two zones with constant thrust F_A and, finally, a ballistic arc at the apogee (Figure 3b).

In the case of circular nominal orbit (e = 0), the efficiency circle (*P*) is reduced to a point ω (Figure 4). /64

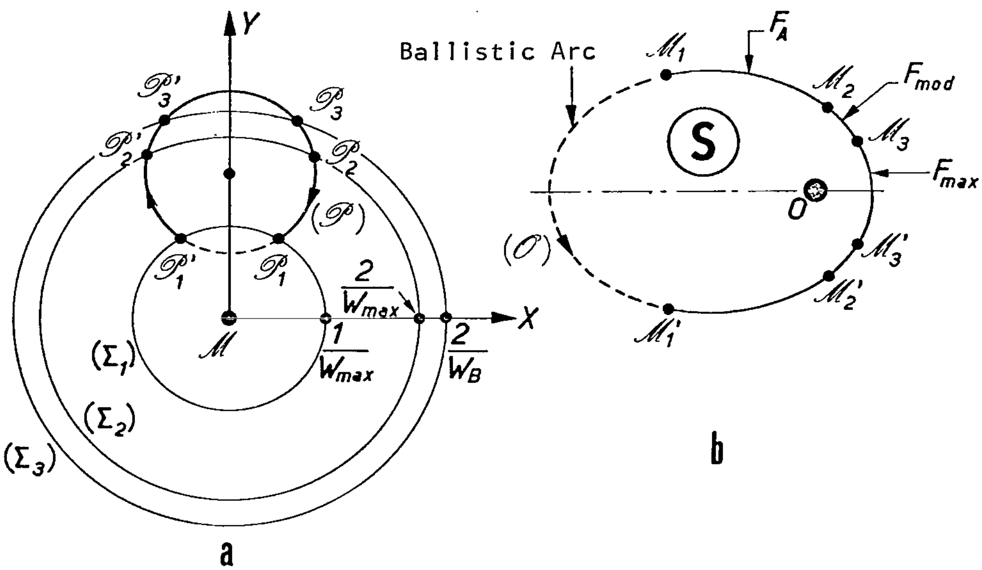


Fig. 3. a) Efficiency Curve; b) Optimal Modulation of the Thrust (System S).

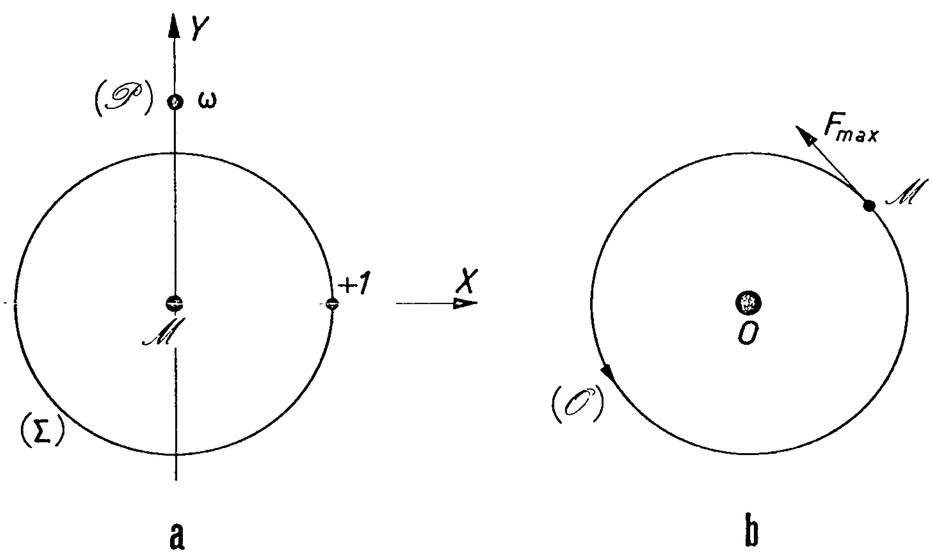


Fig. 4. a) Efficiency Curve → point ω; b) Optimal Thrust (System S_1).

For a propulsion system (S_2), thrust acceleration is constant, horizontal and equal to $\vec{\gamma} = \overrightarrow{M\omega}$.

For a propulsion system (S_1), the maximal thrust is applied constantly if ω is outside the circle (Σ). If ω is on (Σ)we again find the singular case of the type I bis pointed out in § I,3.2.6. (Figure 5). The thrust is horizontal but seldom limited to $0 \leq F \leq F_{max}$. The corresponding degeneracy will be studied later.

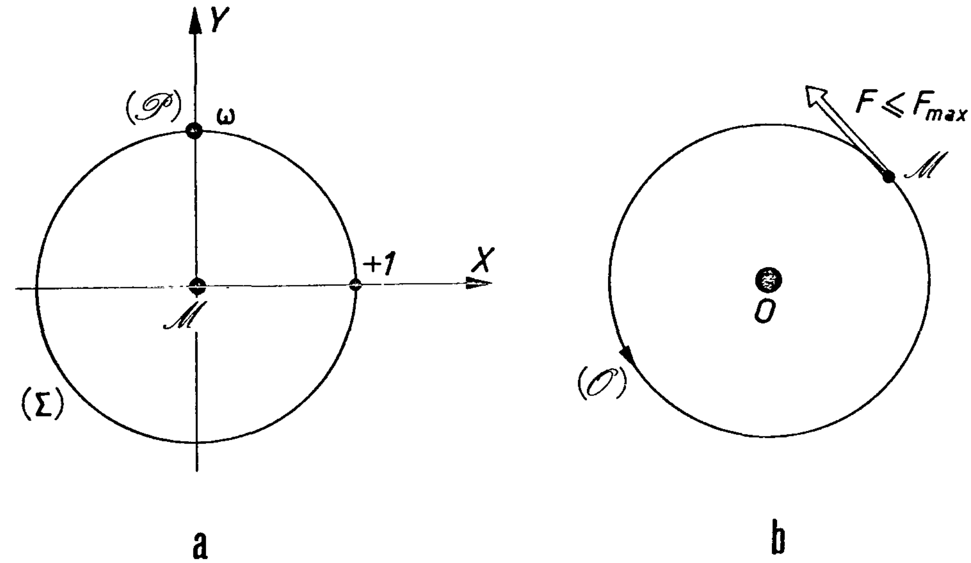


Fig. 5. a) Efficiency Curve → Point ω; b) Optimal Thrust (System S_1). Singular Case I bis.

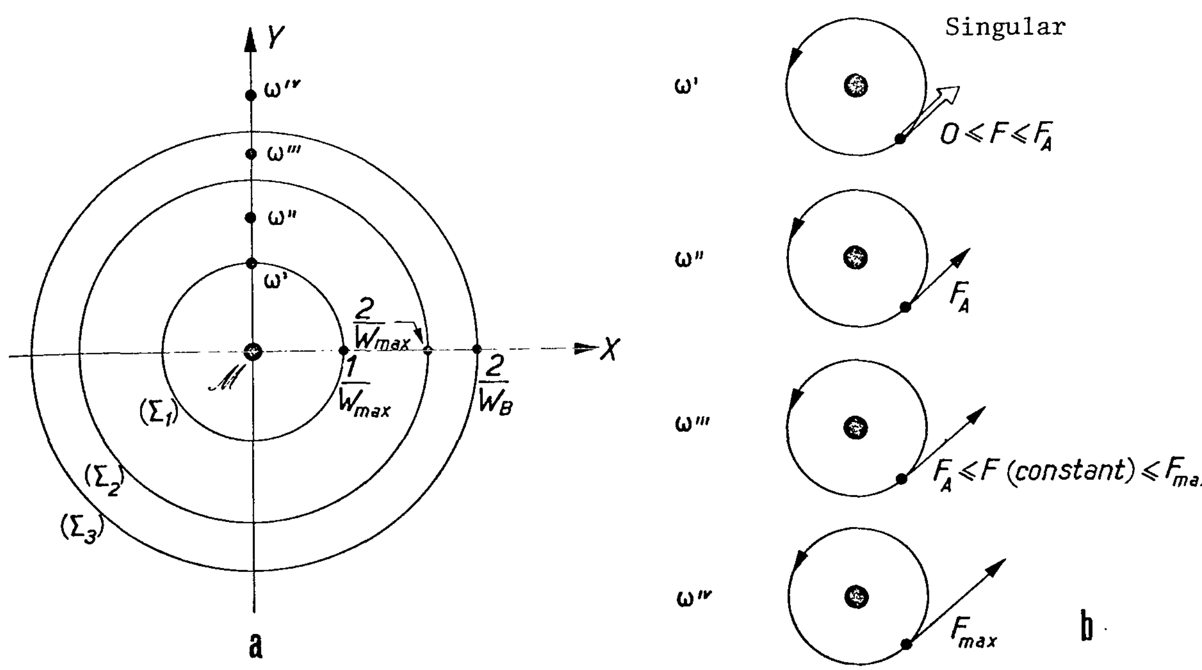


Fig. 6. a) Efficiency Curve → Point ω; b) Optimal Thrust (System S).

Finally for a propulsion system (S), the thrust law depends on the position of ω in relation to the circles with center M and radii $\frac{1}{W_{max}}, \frac{2}{W_{max}}, \frac{2}{WB}$ (Figure 6). /65

II,1.3. Optimal "Dilatation" of the Orbit. Consumption.

II,1.3.1. Elliptical case. Whole number of revolutions.

In order to avoid discussing it as a function of u_0, we shall suppose that the transfer takes place in a whole number of revolutions ($\Delta L = 2N\pi$).

We know that then there is no variation $\Delta\alpha$ and $\Delta\beta$ induced by the variation Δa in the case of system (S_2). For the system (S_1), there is a variation $\Delta\alpha$ induced from the eccentricity, but there is no rotation ($\Delta\beta/e$) induced from the orbit in its plane (See Chapter I,4.).

II,1.3.1.1. System (S_2).

The reduced variation is given by equation (I,4-3):

$$|\zeta| = \frac{|\Delta a|}{\sqrt{2}\, F_{max}\, |\Delta m/\Delta L|} = \sqrt{\langle G_{aa} \rangle} = 2 \text{ (indépendent of } e\text{)}. \tag{3}$$

II,1.3.1.2. System (S_1).

The variation Δa is given by equation (I,3-66):

$$\Delta a = G_{aa}\, p_a = 2F_{max}\,(\text{sign } p_a) \int_{u_0}^{u_f} U(\Theta) \sqrt{1 - e^2 \cos^2 u}\, du$$

or: /66

$$\lambda = \frac{\Delta a}{2N\pi F_{max}} = \frac{2\,(\text{sign } p_a)}{\pi} \begin{cases} E\left(e, \frac{\pi}{2}\right) - E\left(e, \frac{\pi}{2} - u_1\right) \text{ si } 0 \leqslant u_1 \leqslant \frac{\pi}{2} \\ E\left(e, \frac{\pi}{2}\right) + E\left(e, u_1 - \frac{\pi}{2}\right) \text{ si } \frac{\pi}{2} \leqslant u_1 \leqslant \pi \end{cases} \tag{4}$$

where $E(k, \phi)$ is the elliptical integral of the second species of modulus K and of argument ϕ and u_1 the commutation point. From this we easily deduce: sign p_a = sign Δa.

Consumption is given by (I,3-68):

$$\Delta C = \int_{u_0}^{u_F} F_{max}\, U(\Theta)\, r\, du = 2N F_{max}\,(u_1 - e \sin u_1)$$

whence

$$\lambda_c = \frac{\Delta C}{2N\pi\ F_{max}} = \frac{W/\Delta m/}{2N\pi F_{max}} = \frac{u_1 - e\sin u_1}{\pi}. \tag{5}$$

The variation Δa and consumption $|\Delta m|$ are therefore given in a *parametric* form as a function of the magnitude of the maximal thrust arc (u_1).

Figure 7 shows the evolution of the specific dilatation $|\nu| = |\Delta a|/\Delta C = |\lambda|/\lambda c$ and of $|\lambda|$ as a function of u_1 for $e = 1/\sqrt{2}$:

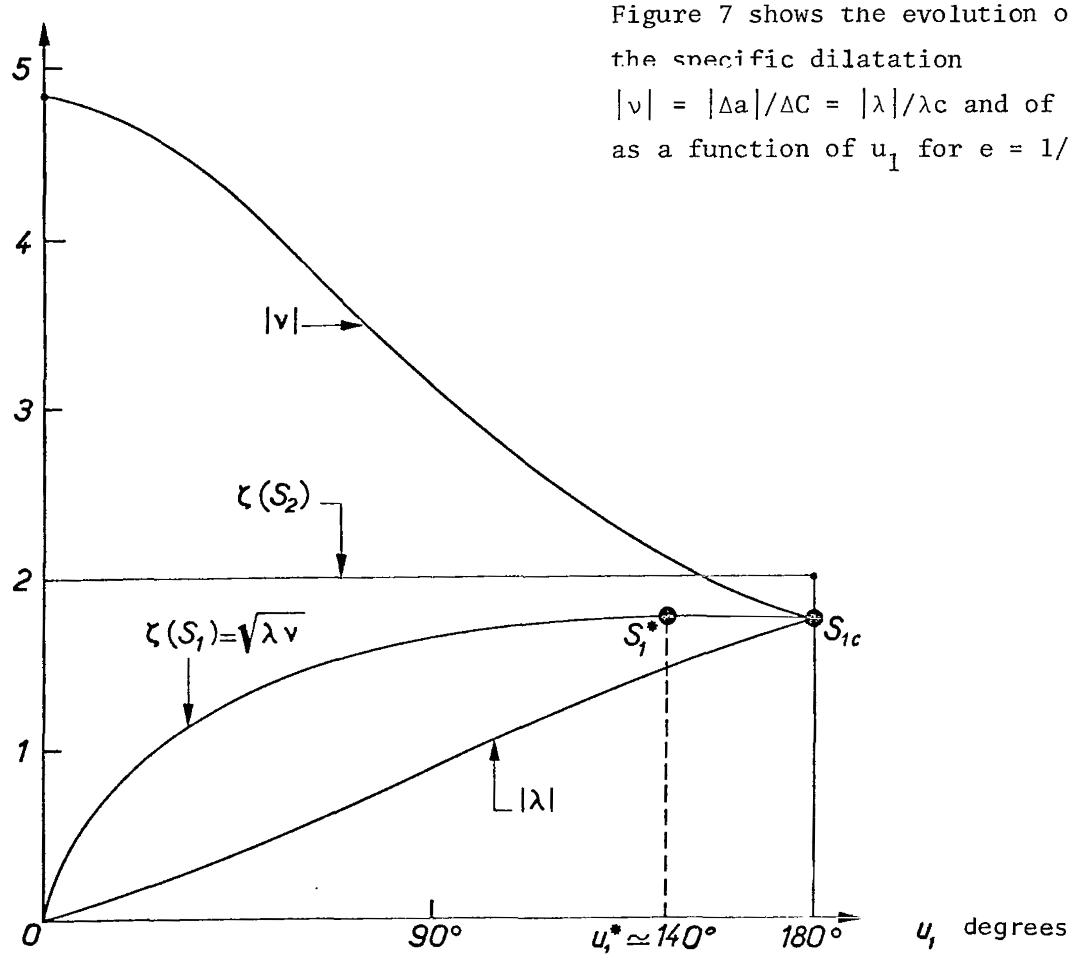


Fig. 7. Comparative Performances of Propulsion Systems. ($e = 1/\sqrt{2}$).

The specific dilatation $|\nu|$ is maximal for the impulsional solutions ($u_1 = 0$), which hold for $|\lambda| = 0$, i.e. a maximal thrust acceleration (in relationship to local gravity), which is very large compared to the relative dilatation $|\Delta a|/a$ of the semi-major axis to be achieved *by revolution*.

This maximal specific dilatation is equal to: /67

$$|\nu|_{max} = \lim_{u_1\to 0} |\nu| = \lim_{u_1\to 0} \frac{2\sqrt{1-e^2\cos^2 u_1}}{1-e\cos u_1} = 2\sqrt{\frac{1+e}{1-e}}. \tag{6}$$

Or, for $e = 1/\sqrt{2}$, $|\nu|_{max} = 4.83$. It corresponds to the application of thrust to the point (perigee) of maximal efficiency.

Beyond the value:

$$|\lambda|_{max} = \frac{4E(e)}{\pi} \tag{7}$$

(where E(k) represents the complete elliptical integral of the second species of modulus k) corresponding to $u_1 = \pi$, i.e. to continuous thrust, transfer is no longer achievable: F_{max} is not sufficient or N is not sufficient. The corresponding value of $|\nu|$ is:

$$|\nu|_{min} = |\lambda|_{max} = \frac{4E(e)}{\pi}. \text{ Here, for } e = 1/\sqrt{2}, |\nu|_{min} = |\lambda|_{max} = 1,72.$$

The fact of not being able to choose N and F_{max} correctly in order to achieve a determined dilatation can lead to a penalty in specific consumption (therefore in consumption) as far as:

$$\frac{\frac{1}{1,72} - \frac{1}{4,83}}{\frac{1}{4,83}} \simeq 1,8 \quad \text{or} \quad 180\ \%.$$

II,1.3.1.3. Comparison between systems (S_1) and (S_2).

In figure 7 is shown the reduced dilatation

$$|\zeta| = \frac{|\Delta a|}{\sqrt{2 P_{max} / \Delta m / \Delta u}} = \sqrt{\lambda\nu}$$

in the case of system (S_1). Evidently it is always lower than that obtained by the system (S_2) ($|\zeta| = 2$).

For the system (S_1) and in the case of $e = 1/\sqrt{2}$, $|\zeta|$ increases from zero, passes through a very flat maximum for $u_1 = u_1^* \simeq 140°$, which corresponds to the optimal choice of the ejection velocity ($\dot{W} = \dot{W}^*$), then decreases slightly to

$u_1 = 180°$. The optimal (S_1^*) system ($W = W^*$) and the system (S_{1c}) corresponding to the continuous application of thrust in this case have completely comparable performances.

The non-modulation of the ejection velocity leads to a penalty in reduced dilatation at least equal to:

$$\frac{\gamma(S_2) - \gamma(S_1^*)}{\gamma(S_2)} = \frac{2 - 1,74}{2} \simeq 0,13 \tag{8}$$

or about 26% of the propulsive consumption.

NOTE: For $0 \leq e \leq e_{lim}$ (root of the equation $\frac{E\left(e, \frac{\pi}{2}\right)}{\pi} = \sqrt{\frac{1-e}{1+e}}$, $u_1^* = 180°$) and $(S_1^*) \equiv (S_{1c})$.

II,1.3.1.4. System (S). /68

The optimal solution depends on the single parameter p_a, (or rather on parameter $W_B p_a$).

The commutation points are given by:

$$\left.\begin{aligned}
\cos u_1 = \cos u_1' &= \frac{1}{e}\,\frac{1 - 4p_a^2 W_{max}^2}{1 + 4p_a^2 W_{max}^2} = \frac{1}{e}\,\frac{1-(4p_a^2 W_B^2/f^2)}{1+(4p_a^2 W_B^2/f^2)} \\
\cos u_2 = \cos u_2' &= \frac{1}{e}\,\frac{1 - p_a^2 W_{max}^2}{1 + p_a^2 W_{max}^2} = \frac{1}{e}\,\frac{1-(p_a^2 W_B^2/f^2)}{1+(p_a^2 W_B^2/f^2)} \\
\cos u_3 = \cos u_3' &= \frac{1}{e}\,\frac{1 - p_a^2 W_B^2}{1 + p_a^2 W_B^2}
\end{aligned}\right\} \tag{9}$$

with

$$f = \frac{F_A}{F_{max}} < 1.$$

The dilatation of the semi-major axis is:

$$\lambda = \frac{\Delta a}{2N\pi F_{max}} = \frac{2(\text{sign } p_a)}{\pi}\left\{f\int_{u_2}^{u_1}\sqrt{1-e^2\cos^2 u}\,du + W_B|p_a|\Big[u + e\sin u\Big]_{u_3}^{u_2} + \int_0^{u_3}\sqrt{1-e^2\cos^2 u}\,du\right\} \tag{10}$$

and the consumption:

$$\lambda_c = \frac{W_B \, |\Delta m|}{2N\pi F_{max}} = \frac{1}{\pi} \left\{ f^2 \Big[u - e \sin u \Big]_{u_2}^{u_1} + p_a^2 \, W_B^2 \Big[u + e \sin u \Big]_{u_3}^{u_2} + \Big[u - e \sin u \Big]_0^{u_3} \right\} \quad 11)$$

The elliptical integrals occurring in (10) have already been calculated for the system (S_1).

Figure 8 shows the evolution of the commutation points u_1, u_2, u_3 and of the "specific consumption" $1/|\nu|$ as a function of $|\lambda|$ for $e = 1/\sqrt{2}$ and $f = \frac{F_A}{F_{max}} = 0.1$.

From A to B, the thrust zone with $F = F_A$ extends around the perigee (u_1 increases from zero).

At the perigee in B there appears a modulated thrust zone ($F_A \leq F \leq F_{max}$) which gradually stretches out (u_2 increases from zero).

In C the ballistic zone disappears ($u_1 = 180°$).

In D the constant thrust zone $F = F_A$ disappears.

From D to E the thrust is modulated along the orbit as if for a propulsion system (S_2). Let us note that then:

$$\frac{1}{|\nu|} = \frac{|\lambda|}{\zeta^2} = \frac{|\lambda|}{4}$$

therefore DE is a straight line segment (whence the interest in considering $\frac{1}{|\nu|}$ and not $|\nu|$).

In E appears a maximal thrust zone F_{max} which extends around the perigee (u_3 increases from zero).

Finally, beyond F corresponding to continuous maximal thrust ($u_3 = 180°$), /69
transfer is impossible.

In Figure 8 has also been shown the specific consumption referring to the system (S_1) of the same installed power and of the same maximal thrust.

This specific consumption is evidently stronger than in the case of system (S), since the command domain is reduced.

On the other hand, the specific consumption ($\frac{1}{|\nu|} = |\lambda|/4$) referring to the system (S_2) of the same installed power is weaker than in the case of the system (S) (or equal, on the segment DE) since the command domain is enlarged.

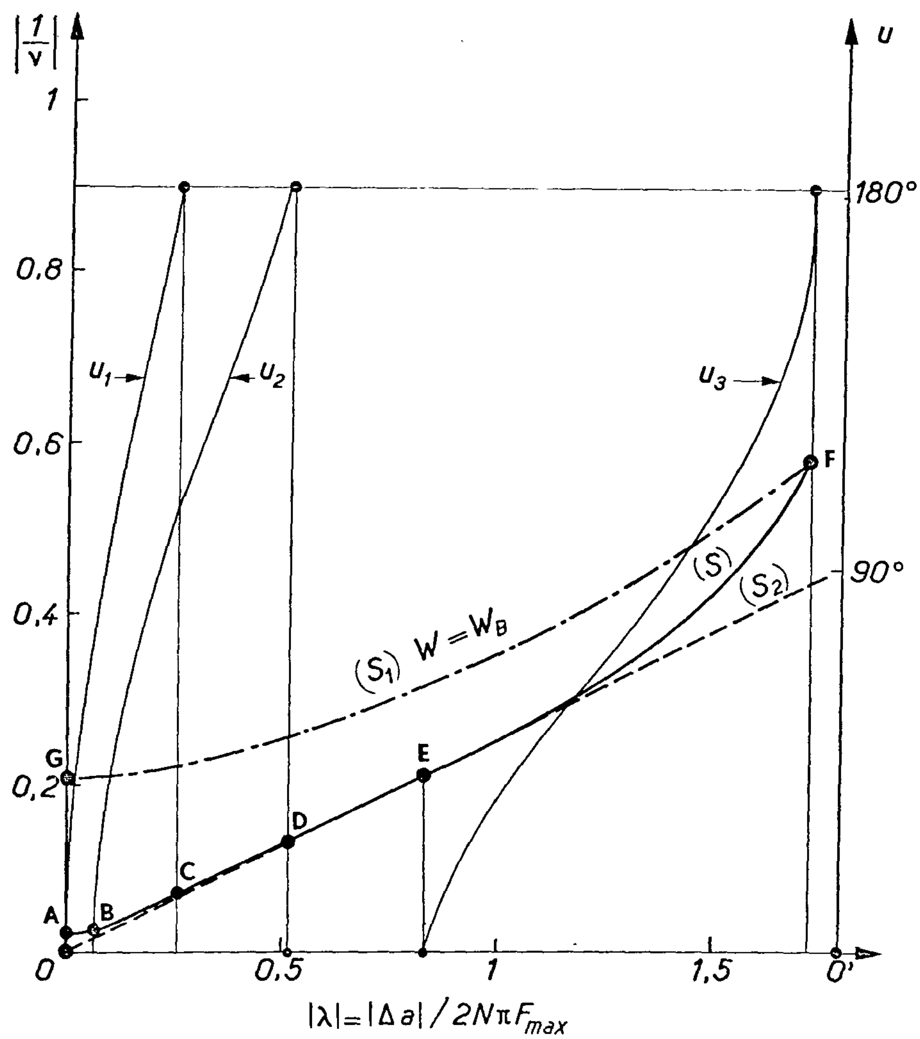


Fig. 8. Performance of System (S). $(e = 1/\sqrt{2}, f = F_A/F_{max} = 0{,}1)$.

Let us note that since the specific consumption of an impulsional transfer is independent of the magnitude of the thrust utilized, /70

$$\frac{W_B \, |\Delta m|_G}{|\Delta a|} = \frac{W_{max} \, |\Delta m|_A}{|\Delta a|}$$

and since the ejection velocity fitting the definition of $|\nu|$ is always W_B,

$$OA = \frac{W_B \, |\Delta m|_A}{\Delta_a} = \frac{W_B}{W_{max}} \; \frac{W_B \, |\Delta m|_G}{|\Delta_a|} = \frac{W_B}{W_{max}} \; OG = f \; OG.$$

Figure 9 shows the evolution of the optimal thrust in the case $|\lambda| = 1.19$ for $e = 1/\sqrt{2}$ and the propulsion systems (S_1), (S_2) and (S) (f = 0.1).

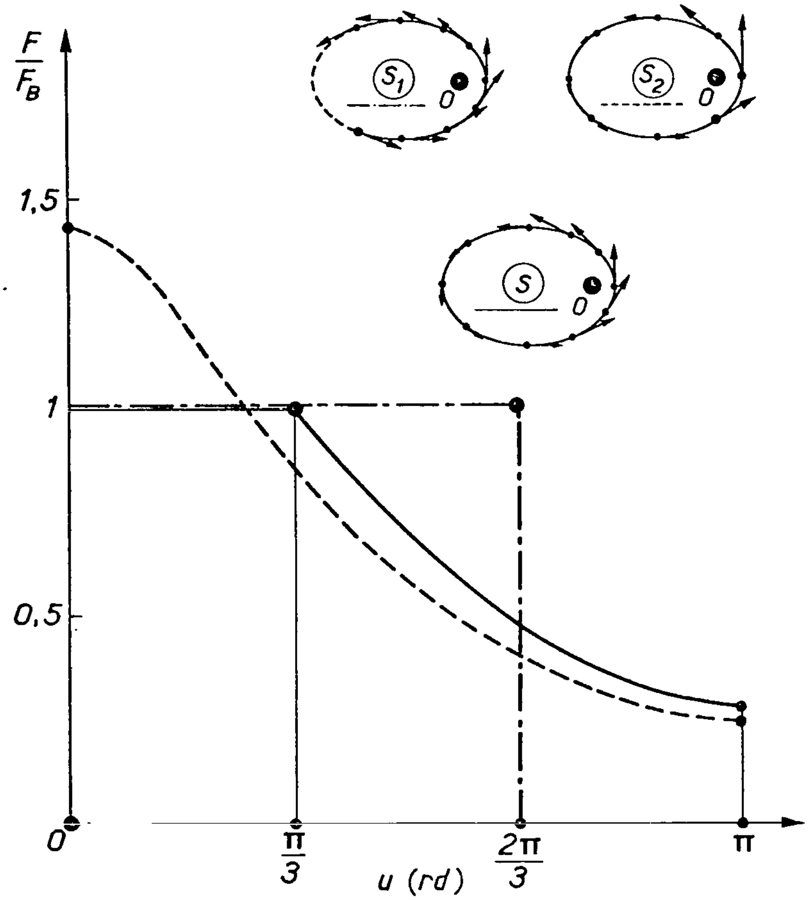


Fig. 9. Comparison of the Thrust Laws of the Propulsion Systems.

$\left(e = \frac{1}{\sqrt{2}} \ ; f = 0{,}1 \ ; |\lambda| = 1{,}19\right)$

II,1.3.2. Circular case (e = 0) -- Any angle of transfer. /71

We take the bisectrix of the transfer angle as the axis of reference $\vec{Ox}\,(L_0 = -\Delta L/2 \ ; L_f = +\Delta L/2)$.

II,1.3.2.1. System (S_2).

The thrust is tangential and constant, and equal to

$$F = \frac{|\Delta_a|}{2\,\Delta L} \tag{12}$$

and the greater ΔL is, the weaker it is.

II,1.3.2.2. System (S_1).

If $F_{max} < \frac{|\Delta a|}{2\Delta L}$ transfer is impossible.

If $F_{max} = \frac{\Delta a}{2\Delta L}$ transfer consists of always thrusting tangentially with maximal thrust.

If $F_{max} > \frac{|\Delta a|}{2\Delta L}$ the solution degenerates (singular case of type I bis). The thrust law referring to any degenerated solution whatever is obtained by dividing the ficticious "mass" $|\Delta a|/2$ on the transfer arc $\widehat{M_0 M_f}$ with the "linear density" $F(L)$ $[0 \leq F(L) \leq F_{max}]$. Thrust is tangential, forward if $\Delta a > 0$, backward if $\Delta a < 0$, and with modulus $F(L)$.

Of the possible distributions, this great degeneracy will permit those which will assure the given variations of certain perameters of the orbit other than a to be chosen (see § II,3.3.2.5.).

II,1.4. Conclusion.

This first particular case, very simple, has been able to be studied in rather unlimited hypotheses and even for the propulsion system (S).

The other cases where a single variation is imposed are more delicate but will be able to be approached in the same manner.

II,2. OPTIMAL INFINITESIMAL ROTATION OF THE PLANE OF THE ORBIT

II,2.1. Introduction.

Here it is a question of realizing optimal rotation $\vec{\vec{\Delta j}}$ of the plane of the orbit (O) around an axis contained in its plane, possible variations induced from other elements being considered as indifferent (Figure 1).

The only components of the kinematic adjoint P not necessarily zero are p_ξ and p_η.

In Chapter I,4. we have already obtained the following results: /72

there is no variation induced from the other elements and *for the propulsion system* (S_2), there is uncoupling between the problem of the optimal rotation of the plane of the orbit and the problem of the modifications of the orbit in its plane, i.e. the optimal solution which we are going to find concerning the rotation of the plane alone will be able to be superposed on the optimal solution concerning the modifications of the orbit in its plane, if these modifications are likewise imposed.

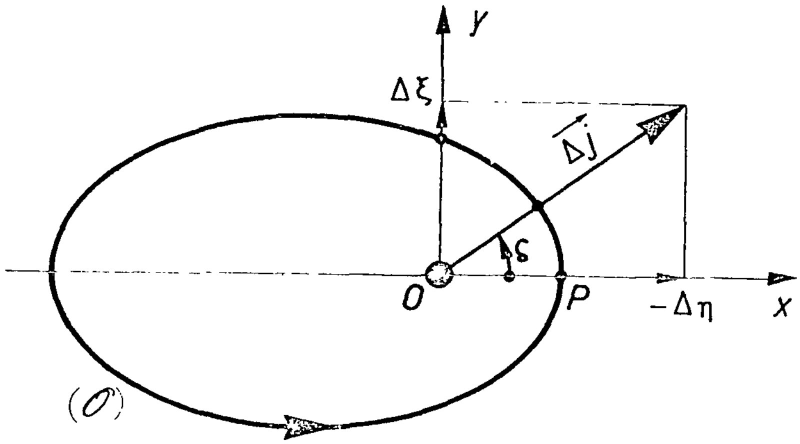


Fig. 1. Infinitesimal Rotation of the Plane of the Orbit.

II,2.2. Optimal Thrust.

Equation (I,3 - 41) is written here:

$$\vec{p_V} = \frac{\vec{p_z}}{h} \wedge \vec{r} = \frac{p_\xi y - p_\eta x}{h} \vec{z} \qquad \left(h = \sqrt{1-e^2}\right) \tag{1}$$

and shows that the vector of efficiency $\vec{p}_V$ (giving the optimal thrust direction $\vec{D}$) is *normal* to the plane of the orbit. It is zero when $\vec{r}$ is colinear with $\vec{p}_z$.

For a normalization condition of the adjoint such that $|\vec{P}_V| << 1$, *the directrix orbit* (C_P) has as elements (equation I,3, 47-49)

$$\left.\begin{aligned} \vec{h}_{\mathcal{P}} &= \vec{h} + \vec{Dh} = \vec{h} + \vec{p_z} \wedge \vec{z} \\ \vec{e}_{\mathcal{P}} &= \vec{e} + \vec{De} = \vec{e} + \frac{\vec{p_z}}{h} \wedge \vec{e} \\ a_{\mathcal{P}} &= a \end{aligned}\right\} \tag{2}$$

The plane of this ellipse (O_P) is deduced from the plane of ellipse (O) by the infinitesimal rotation $\vec{p}_z/h$ (Figure 2).

In this particular problem of the optimal rotation of the plane of the orbit, the locus of the extremity P of the vector $\vec{MP} = \vec{p}_V$ in the absolute /73 space Oxyz is still an ellipse, even if the normalization condition of the adjoint is no longer such that $|\vec{\vec{p}}_V|$ is << 1. This ellipse (E) is the intersection of the elliptical straight cylinder of generatrices parallel to $\vec{\vec{Oz}}$ based on (O) and of the plane:

$$z = \frac{y p_\xi - x p_\eta}{h} \qquad \text{(Fig. 3)} \qquad (3)$$

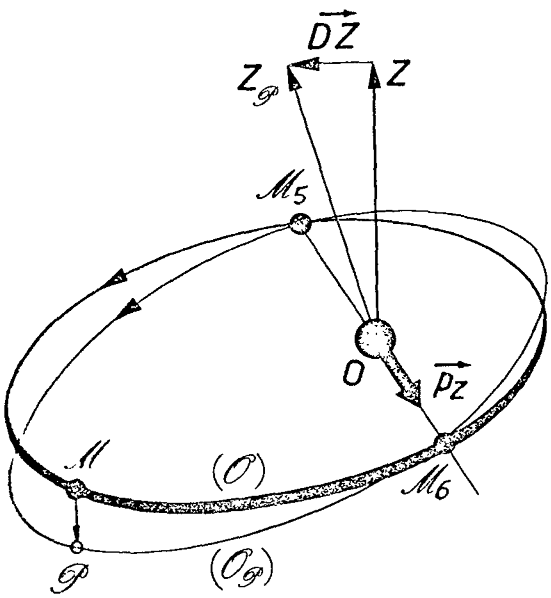


Fig. 2. Directrix Orbit.

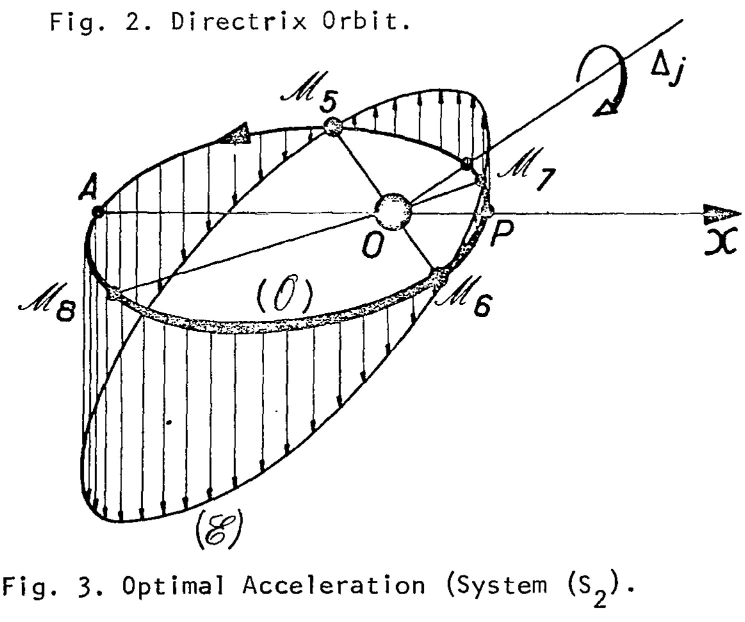


Fig. 3. Optimal Acceleration (System (S_2).

II,2.2.1. Propulsion System (S_2).

The optimal thrust acceleration $\vec{\gamma} = \vec{\vec{p}}_V$ is alternatively oriented upward $(z > 0)$ in one half turn, then downward $(z < 0)$ in a half turn (Figure 3). γ is absolutely maximum at the extremity most distant from zero of the congugated diameter M_7M_8 of M_5M_6 in reference to (O) and relatively maximum at the other extremity.

II,2.2.2. Propulsion System (S_1). /74

The commutation points M_1, M_2, M_3, M_4 are optained by cutting the ellipse (E) by the planes $z = \pm 1$ $(|\vec{\vec{p}}_V| = 1)$. Therefore there is a maximum per revolution of one or two maximal thrust arcs (Figure 4).

When there are two thrust arcs, the cords of the thrust arcs are parallel and equidistant from zero. (OH' = OH'' in Figure 5).

In the case of propulsion system (S_1) it is convenient to define the solution, no longer by the parameters p_ξ and p_η, but by the parameters α and S indicated in Figure 5, introducing the principal circle (C) of the orbit (O) and such that:

$$\begin{cases} \cos\alpha = -s p_\eta / h \\ \sin\alpha = s p_\xi \end{cases} \tag{4}$$

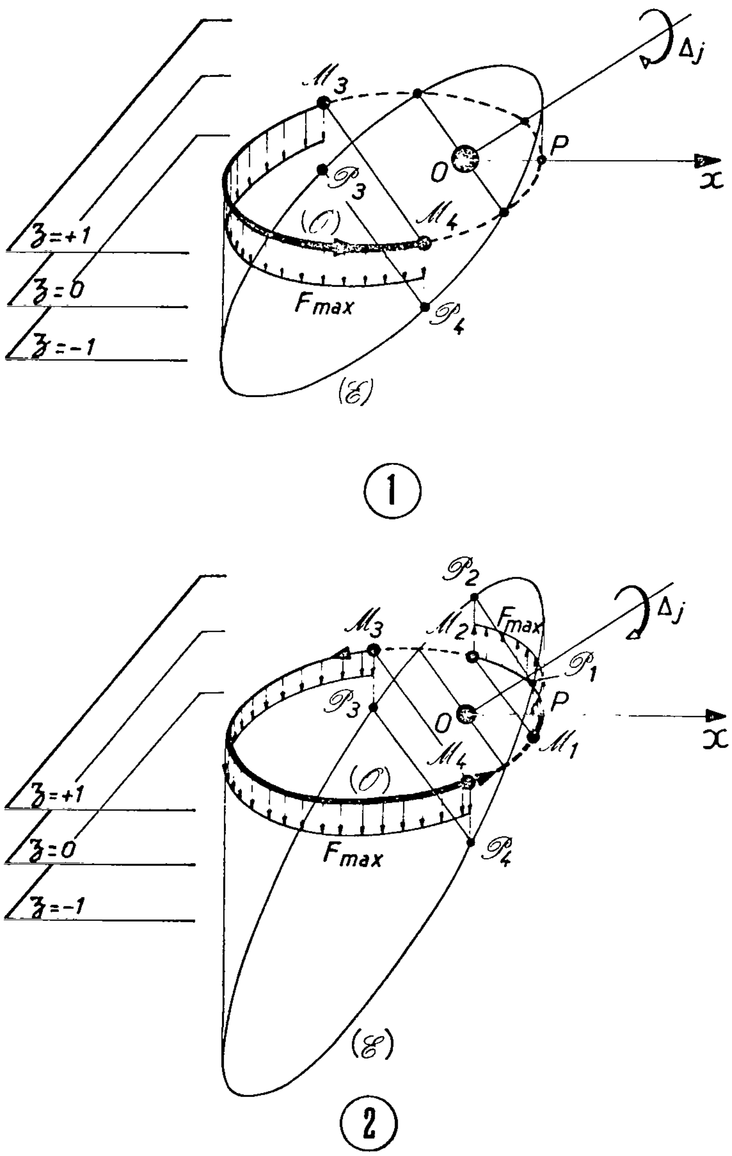


Fig. 4. Optimal Thrust (System S_1).
1. One Maximal Thrust Arc.
2. Two Maximal Thrust Arcs.

It is easy to see that the study for $0 \leq \alpha \leq \frac{\pi}{2}$ is sufficient./75

Then the commutation points given as a function of α and s by:

$$\cos(u-\alpha) = e\cos\alpha \pm s \tag{5}$$

whence

$$\begin{cases} u_1 = \alpha - \beta' \\ u_2 = \alpha + \beta' \\ u_3 = \alpha + \beta'' \\ u_4 = \alpha - \beta'' \end{cases} \tag{6}$$

with

$$\left.\begin{aligned} \beta' &= \operatorname{Arc}\cos(e\cos\alpha + s) \\ \beta'' &= \operatorname{Arc}\cos(e\cos\alpha - s) \end{aligned}\right\} \tag{7}$$

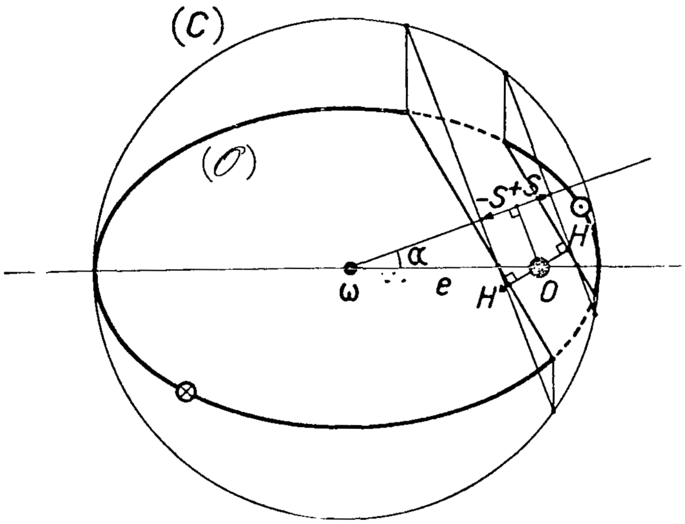
(see Figure 6).

According to the position of point S of the polar coordinates α and s in the plane (S), we get (Figure 7):

zone (1): a maximal thrust arc F_{max} per revolution.

zone (2): two maximal thrust arcs per revolution forming a "couple".

The frontiers (Γ') and (Γ'') are some of Pascal's great wheels:



$OH' = OH''$

⊙ Upward Thrust

⊗ Downward Thrust

Fig. 5. Definition of the Parameters α and s.

$$\left.\begin{array}{ll}(\Gamma') : & \cos\beta' = e\cos\alpha + s = +1 \\ (\Gamma'') : & \cos\beta'' = e\cos\alpha - s = -1\end{array}\right\} \quad (8)$$

The point 0 corresponds /77
to the continuous application of thrust and the point

$$S_1\left(\alpha = \frac{\pi}{2},\ s = 1\right)$$

to the bi-impulsional solutions (two impulses in contrary directions at the peaks B and B' of the minor axis) which will be studied later in detail.

In the plane (S), the lines $u = C^{te}$ are circles passing through 0 and tangent to (Γ') or (Γ'') of polar equation:

$$\pm s = D(u)\cos[\alpha - W(u)] \qquad (9)$$

with

$$\begin{cases} \cos W(u) = \dfrac{\cos u - e}{D} \\ \sin W(u) = \dfrac{\sin u}{D} \end{cases}$$

$$D(u) = (1 - 2e\cos u + e^2)^{1/2}$$

which permits getting the position of the commutation points immediately when α and s are known.

II,2.3. Optimal Rotation. Consumption.

II,2.3.1. Particular case of the optimal rotation of a near-circular orbit ($e \leq$ order $\varepsilon \ll 1$) for any transfer angle ΔL.

$\vec{Ox}$ is taken according to the axis of symmetry of the transfer arc $\widehat{M_0M_f}$.

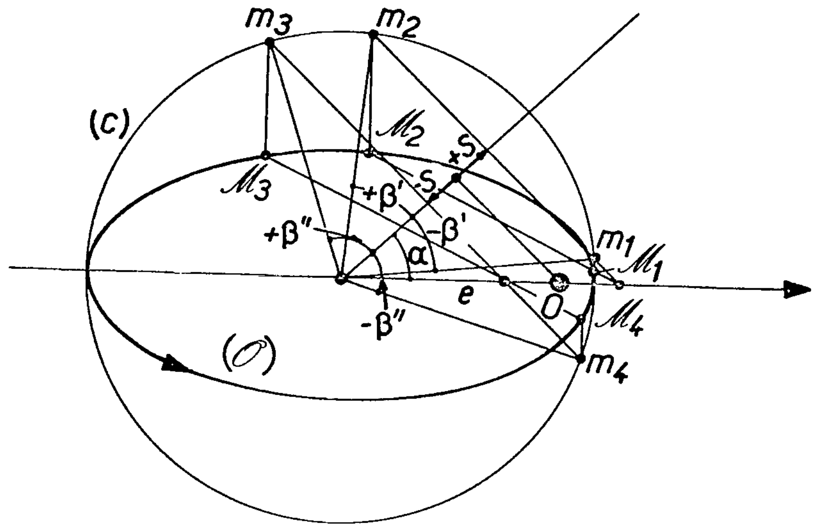
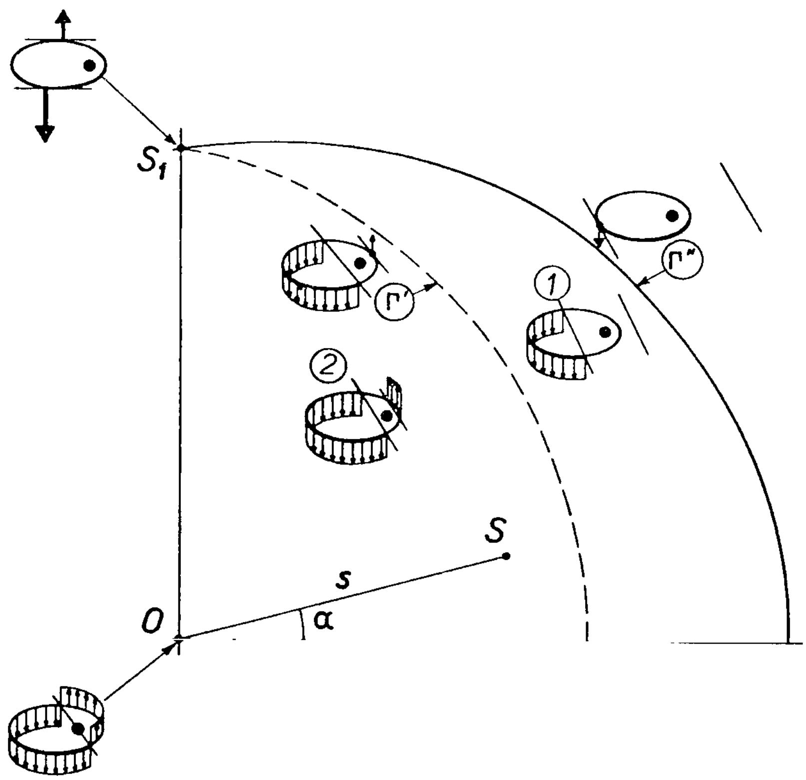


Fig. 6. Definition of the Parameters β' and β''. /76



e Given

Fig. 7. Nomographic Chart in Polar Coordinates α, s. (e = 0.2).

II,2.3.1.1. Propulsion System (S_2).

In § I,4.1.3. we saw that there is uncoupling between the rotation $-\Delta\eta$ around $\overrightarrow{Ox}$ and the rotation $\Delta\xi$ around $\overrightarrow{Oy}$. The study of the rotation $-\Delta\eta$ alone (Figure 8), then the rotation $\Delta\xi$ alone (Figure 9), is enough to solve the problem.

The reduced rotation in both cases is, respectively:

$$|\zeta_\eta| = \frac{|\Delta\eta|}{\sqrt{2P_{max}/\Delta m/\Delta L}} = \sqrt{\langle G_{\eta\eta}\rangle} \tag{10}$$

and

$$|\zeta_\xi| = \frac{|\Delta\xi|}{\sqrt{2P_{max}/\Delta m/\Delta L}} = \sqrt{\langle G_{\xi\xi}\rangle} \tag{11}$$

where

$$\langle G_{\eta\eta}\rangle = \frac{G_{\eta\eta}}{\Delta L} = \frac{1}{2}\left(1 + \frac{\sin\Delta L}{\Delta L}\right) \tag{12}$$

$$\langle G_{\xi\xi}\rangle = \frac{G_{\xi\xi}}{\Delta L} = \frac{1}{2}\left(1 - \frac{\sin\Delta L}{\Delta L}\right) \tag{13}$$

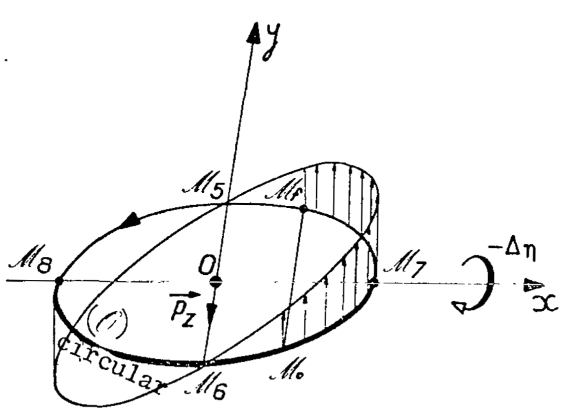


Fig. 8. Rotation Around the Axis of Symmetry of the Transfer Arc (e = 0, System S_2).

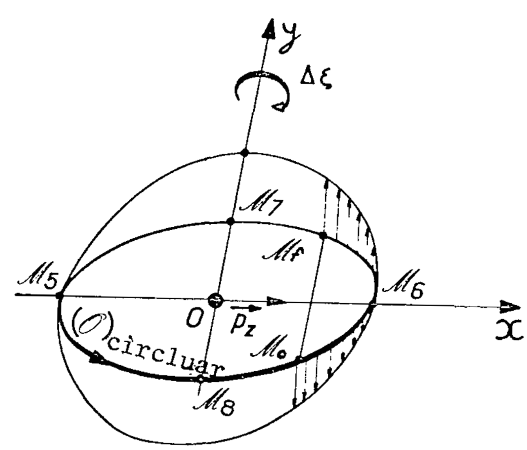


Fig. 9. Rotation Around the Perpendicular to the Axis of Symmetry of the Transfer Arc (e = 0, System (S_2) /78

The reduced rotations $|\zeta_\eta|$ and $|\zeta_\xi|$ are equal to $1/\sqrt{2}$ for $\Delta L = 2N\pi$ (Figure 12) and tend toward this value after a few oscillations when $\Delta L \to \infty$ in conformity with the results obtained in § I,4.1.2.

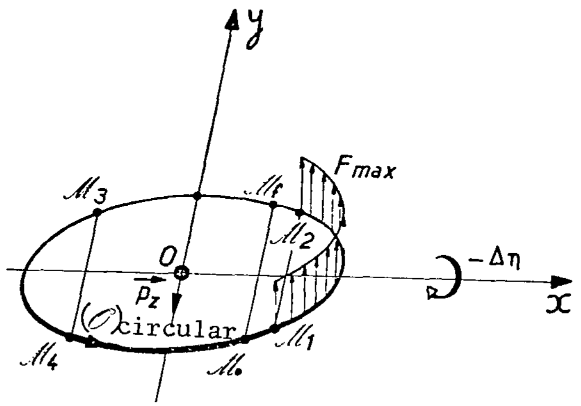


Fig. 10. Rotation Around the Axis of Symmetry ofthe Transfer Arc (e = 0, System S_1).

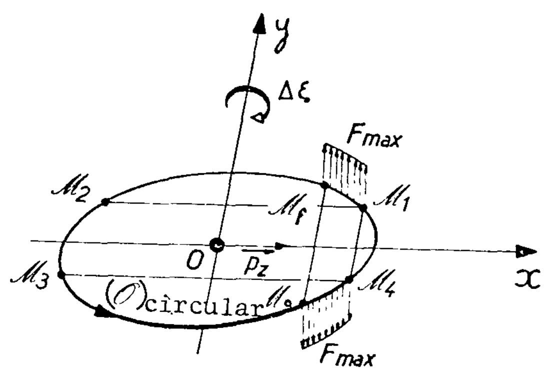


Fig. 11. Rotation Around the Perpendicular to the Axis of Symmetry of the Transfer Arc (e = 0, System S_1).

But it can be noted that these quantities also assume the value $1/\sqrt{2}$ for $\Delta L = (2k + 1)\,\pi$, because the thrust arcs

$$\widehat{\mathcal{M}_6 \odot \mathcal{M}_7 \odot \mathcal{M}_5}$$

and

$$\widehat{\mathcal{M}_5 \otimes \mathcal{M}_8 \otimes \mathcal{M}_6}$$

(Figure 8) or

$$\widehat{\mathcal{M}_8 \otimes \mathcal{M}_6 \odot \mathcal{M}_7}$$

and

$$\widehat{\mathcal{M}_7 \odot \mathcal{M}_5 \otimes \mathcal{M}_8}$$

(Figure 9) have equivalent effects.

The rotation $|\Delta\xi|$ around $\overrightarrow{Oy}$ is much more expensive than rotation $|\Delta\eta|$ around $\overrightarrow{Ox}$ for weak ΔL.

II,2.3.1.2. Propulsion System (S_1).

In § I,4.2.3. we saw that when the rotation $-\Delta\eta$ around $\overrightarrow{Ox}$ (or $\Delta\xi$ around $\overrightarrow{Oy}$) is *alone* imposed, it does not induce any rotation $\Delta\xi$ around $\overrightarrow{Oy}$ (or $-\Delta\eta$ around $\overrightarrow{Oy}$). But there is no *uncoupling* between these two rotations: the successive study of both of these rotations is not enough to resolve the problem by adding the solutions. However, we /79
shall limit ourselves to these two simple cases in order to compare the results obtained with those referring to the propulsion system (S_2).

ROTATION - $\Delta\eta$ ALONE IMPOSED (Figure 10).

In this case $p_\xi = 0$.

The rotation is given by:

$$\lambda_\eta(\Delta L,\, L_2) = \frac{\Delta\eta}{F_{max}\,\Delta L} = \frac{(\text{sign}\; p_\eta)}{\Delta L}\int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{F_{max}}\,|\cos L|\,dL = \tag{14}$$

$$\frac{(\text{sign } p_\eta)}{\Delta L}\begin{cases} 2\sin\bar{L} + 4N\sin L_2 \text{ for } 2N\pi \leqslant \Delta L \leqslant 2L_2 + 2N\pi \\ 2\sin L_2 + 4N\sin L_2 \text{ for } 2L_2 + 2N\pi \leqslant \Delta L \leqslant 2(\pi - L_2) + 2N\pi \\ 4\sin L_2 - 2\sin\bar{L} + 4N\sin L_2 \text{ for } 2(\pi - L_2) + 2N\pi \leqslant \Delta L \leqslant 2(N+1)\pi \end{cases} \quad \text{(14) cont.}$$

with

$$0 \leqslant \bar{L} = \frac{\Delta L}{2} - N\pi \leqslant \pi$$

N is the number of complete turns.

The consumption is given by: /80

$$\lambda_c(\Delta L, L_2) = \frac{\Delta C}{F_{max}\,\Delta L} = \frac{W/\Delta m/}{F_{max}\,\Delta L} = \frac{1}{\Delta L}\int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{F_{max}}\, dL = \tag{15}$$

$$\frac{1}{\Delta L}\begin{cases} 2\bar{L} + 4NL_2 \text{ for } 2N\pi \leqslant \Delta L \leqslant 2L_2 + 2N\pi \\ 2L_2 + 4NL_2 \text{ for } 2L_2 + 2N\pi \leqslant \Delta L \leqslant 2(\pi - L_2) + 2N\pi \\ 2\bar{L} + 4L_2 - 2\pi + 4NL_2 \text{ for } 2(\pi - L_2) + 2N\pi \leqslant \Delta L \leqslant 2(N+1)\pi \end{cases}$$

Whence the reduced rotation :

$$|\zeta_\eta| = \sqrt{\lambda_\eta \nu_\eta} = \frac{|\lambda_\eta|}{\sqrt{\lambda_c}} = |\zeta_\eta(\Delta L, L_2)| \,. \tag{16}$$

If P_{max} is fixed, maximizing $|\zeta_\eta|$ in reference to L_2 (for fixed ΔL and $\Delta\eta$) comes down to seeking the optimal (constant) ejection velocity W* [utilization of propulsion system (S_1^*)].

We shall not carry out this calculation and shall limit ourselves to envisaging the case of continuous thrust ($L_2 = \pi/2$) [utilization of propulsion system (S_{1c})].

Then: $\lambda_c = 1$ and

$$|\zeta_\eta|_c = |\lambda_\eta| = \frac{1}{\Delta L}\begin{cases} 2\sin\bar{L} + 4N \quad si\ 2N\pi \leqslant \Delta L \leqslant (2N+1)\pi \\ 4 - 2\sin\bar{L} + 4N \quad si\ (2N+1)\pi \leqslant \Delta L \leqslant 2(N+1)\pi \end{cases} \tag{17}$$

The evolution of this parameter is traced on Figure 12 in order to show the penalty due to the non-modulation of ejection velocity W. After a few oscillations $|\zeta_\eta|$ tends toward $2/\pi$ (instead of $1/\sqrt{2}$ for a modulated ejection velocity), a value which it still has anyway for $\Delta L = k\pi$. The penalty is negligible for $\Delta L \ll \pi$, but reaches

$$\frac{\Delta|\zeta|}{|\zeta|} = \frac{\frac{1}{\sqrt{2}} - \frac{2}{\pi}}{\frac{1}{\sqrt{2}}} \simeq 0,10$$

or about 20% of mass consumption $|\Delta m|$, for $\Delta L = \{^{k\pi}_{\infty}$.

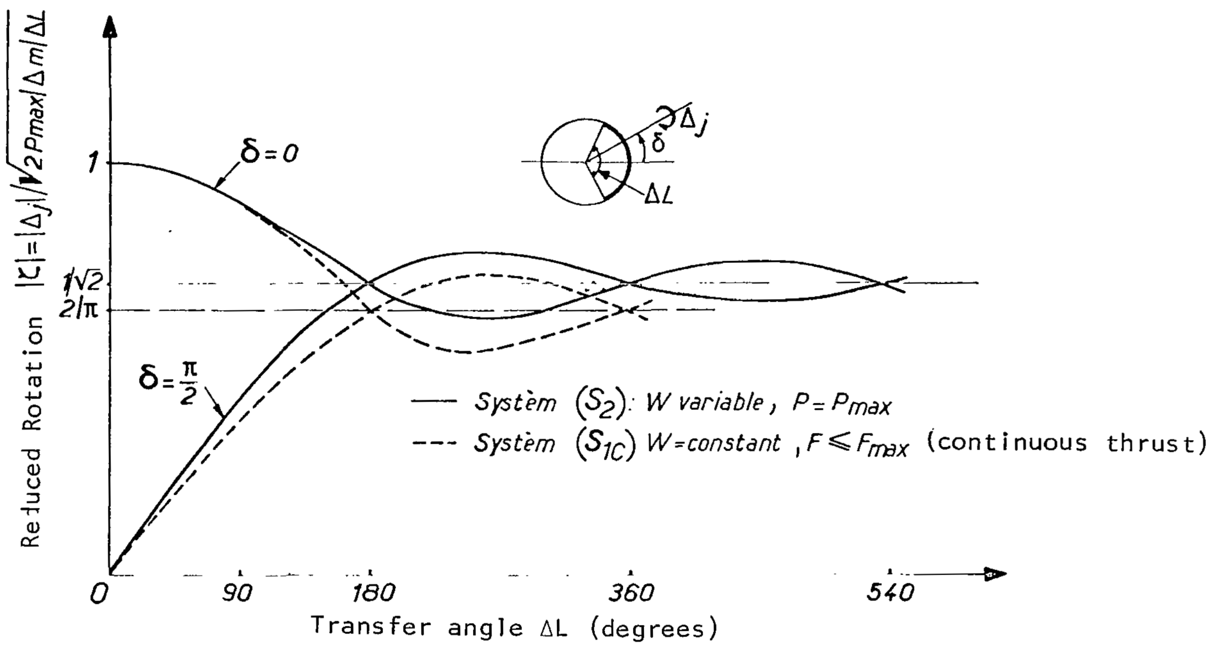


Fig. 12. Comparable Performance of Propulsion Systems (e = 0).

ROTATION $\Delta\xi$ ALONE IMPOSED (Figure 11).

In this case $p_\eta = 0$. We shall only indicate the results:

$$\lambda_\xi(\Delta L, L_1) = \frac{\Delta\xi}{F_{max}\,\Delta L} = \frac{(\text{sign } p_\xi)}{\Delta L} \int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{F_{max}} |\sin L|\, dL = \tag{18}$$

$$\frac{(\mathrm{sign}\ p_\xi)}{\Delta L}\begin{cases} 4N\cos L_1 & \text{for } 2N\pi \leqslant \Delta L \leqslant 2L_1 + 2N\pi \\ 2\cos L_1 - 2\cos\bar{L} + 4N\cos L_1 & \text{for } 2L_1 + 2N\pi \leqslant \Delta L \leqslant 2(\pi - L_1) + 2N\pi \\ 4(N+1)\cos L_1 & \text{for } 2(\pi - L_1) + 2N\pi \leqslant 2(N+1)\pi \end{cases}$$

$$\lambda_c(\Delta L, L_1) = \frac{\Delta C}{F_{max}\,\Delta L} = \frac{W|\Delta m|}{F_{max}\,\Delta L} = \frac{1}{\Delta L}\int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{F_{max}}\,dL =$$

(19) /81

$$\frac{1}{\Delta L}\begin{cases} 2N(\pi - 2L_1) & \text{for } 2N\pi \leqslant \Delta L \leqslant 2L_1 + 2N\pi \\ 2\bar{L} - 2L_1 + 2N(\pi - 2L_1) & \text{for } 2L_1 + 2N\pi \leqslant \Delta L \leqslant 2(\pi - L_1) + 2N\pi \\ 2(N+1)(\pi - 2L_1) & \text{for } 2(\pi - L_1) + 2N\pi \leqslant \Delta L \leqslant 2(N+1)\pi \end{cases}$$

Whence the reduced rotation:

$$|\zeta_\xi| = \sqrt{\lambda_\xi \nu_\xi} = \frac{|\lambda_\xi|}{\sqrt{\lambda_c}} = \left|\zeta_\xi(\Delta L, L_1)\right| \tag{20}$$

and, in the case of continuous thrust ($L_1 = 0$),

$$|\zeta_\xi|_c = |\lambda_\xi| = \frac{1}{\Delta L}(2 - 2\cos\bar{L} + 4N). \tag{21}$$

Figure 12 shows that the rotation $|\Delta\xi|$ is evidently much more expensive than the rotation $|\Delta\eta|$ for $\Delta L \ll \pi$. On the other hand, for $\Delta L = \{^{k\pi}_{\infty}$, the two rotations have equal costs ($|\zeta| = \frac{2}{\pi}$).

The penalty due to non-modulation of ejection velocity W is not negligible for $\Delta L \ll \pi$.

As a matter of fact:

$$|\zeta_\xi|_{(S_2)} = \left[\frac{1}{2}\left(1 - \frac{\sin\Delta L}{\Delta L}\right)\right]^{\frac{1}{2}} \underset{\Delta L\to 0}{\sim} \frac{\Delta L}{2\sqrt{3}}$$

$$|\zeta_\xi|_{(S_{1c})} = \frac{1}{\Delta L}\left(2 - 2\cos\frac{\Delta L}{2}\right) \underset{\Delta L\to 0}{\sim} \frac{\Delta L}{4}$$

whence the penalty:

$$\frac{\Delta|\zeta|}{|\zeta|} \simeq \frac{\frac{1}{2\sqrt{3}} - \frac{1}{4}}{\frac{1}{2\sqrt{3}}} \simeq 1 - \frac{\sqrt{3}}{2} = 0,134$$

or nearly 27% of the mass consumption.

On the other hand, for $\Delta L = \{{}^{k\pi}_{\infty}$, again we find a penalty of 10% on $|\zeta|$, therefore about 20% on $|\Delta m|$.

II,2.3.2. Particular case of a whole number of revolutions ($\Delta L = 2N\pi$).

II,2.3.2.1. Propulsion System (S_2).

The problem was resolved in § I,4.1.2. There is *uncoupling* between the rotation $-\Delta\eta$ around $\vec{Ox}$ and rotation $\Delta\xi$ around $\vec{Oy}$. The study of the rotation $-\Delta\eta$ alone, then of rotation $\Delta\xi$ alone, is enough to resolve the problem.

The corresponding reduced rotations are [see equation (I,4 - 3)]: /82

$$|\zeta_\eta| = \frac{|\Delta\eta|}{\sqrt{2P_{max}|\Delta m/\Delta u}} = \sqrt{\langle G_{\eta\eta}\rangle} = \sqrt{\frac{1+4e^2}{2(1-e^2)}} \quad (22)$$

$$|\zeta_\xi| = \frac{|\Delta\xi|}{\sqrt{2P_{max}|\Delta m/\Delta u}} = \sqrt{\langle G_{\xi\xi}\rangle} = \frac{1}{\sqrt{2}} \quad \text{(independent of e)} \quad (23)$$

The evolution of these parameters as a function of e is given in Figure 13. Note that the greater the eccentricity the less expensive is rotation around the major axis in connection with rotation around the "parameter".

II,2.3.2.2. Propulsion system (S_1).

Although there is no mutual non-induction between the rotations $-\Delta\eta$ around $\vec{Ox}$ and $\Delta\xi$ around $\vec{Oy}$, there is no uncoupling (addition of solutions), with the result that the study of the general case (any angle δ) is necessary.

Integration of the equations of movement on N turns leads to:

$$\lambda_\eta = \frac{\Delta\eta}{2N\pi F_{max}} = -\frac{1}{2\pi h}\int_0^{2\pi} \frac{\pm F}{F_{max}} (\cos u - e)(1 - e\cos u)\, du = \quad (24)$$

$$\frac{1}{2\pi h}\Big[-3e(\beta' + \beta'' - \pi) + 2(1+e^2)\cos\alpha(\sin\beta' + \sin\beta'') - \frac{e}{2}\cos 2\alpha(\sin 2\beta' + \sin 2\beta'')\Big]$$

$$(h = \sqrt{1-e^2})$$

$$\lambda_\xi = \frac{\Delta\xi}{2N\pi F_{max}} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\pm F}{F_{max}} \sin u \,(1 - e\cos u)\, du =$$

$$\frac{1}{2\pi}\left[2\sin\alpha\,(\sin\beta' + \sin\beta'') - \frac{e}{2}\sin 2\alpha\,(\sin 2\beta' + \sin 2\beta'')\right] \qquad (25)$$ /83

$$\lambda_c = \frac{\Delta C}{2N\pi F_{max}} = \frac{W/\Delta m/}{2N\pi F_{max}} = \frac{1}{2\pi}\left[2(\beta' + \pi - \beta'') - 2e\cos\alpha\,(\sin\beta' - \sin\beta'')\right] \qquad (26)$$

where β' and β'' are given in (7) as a function of α and s.

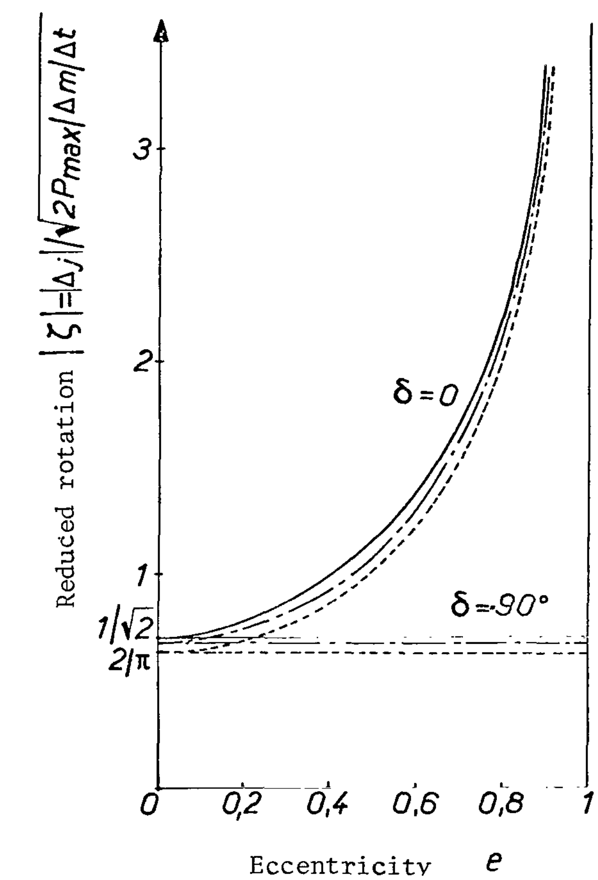


Fig. 13. Contrasted Performance of Propulsion Systems (Whole Numbers in Revolutions).

— System (S_2) --- System (S_{1C}) — System (S_1^{*})

$W var, P = P_{max}$ $W = cte, F \leq F_{max}$ $W = cte = W^{*}_{optimal}$

(Continuous thrust)

Therefore λ_η, λ_ξ and λ_c depend only on the two parameters α and s, that is to say on the polar coordinates of point S in the plane (S). It is understood that β' must be taken as equal to 0 in the expressions in zone (1).

Inversion of the first two equations (calculation of α and s as a function of Δ_η and Δ_ξ in order to introduce these values into the third equation to get an explicit formula giving the consumption as a function of the rotation desired) is possible only in certain particular cases. But these formulae lend themselves very well to a numerical resolution by successive approximations.

This resolution is facillitated by using nomographic charts traced for constant values of eccentricity e in the plane (S) which furnish a first approximation by interpolation.

Figures 14, 15 and 16 show such nomographic charts traced respectively for e = 0, 0.2 and 0.8.

Only the lines

$$\delta = \mathrm{Arc\ tg}\left|\frac{\Delta\xi}{\Delta\eta}\right| = C^{te},$$

$$|\lambda| = \frac{|\Delta j|}{2N\pi F_{max}} = \sqrt{\lambda_\xi^2 + \lambda_\eta^2} = C^{te},$$

characterizing rotation, and $|\nu| = \frac{|\lambda|}{\lambda_c} = \frac{|\Delta j|}{\Delta c} = \frac{|\Delta j|}{W/\Delta m/} = C^{te}$ (specific rotation), have been represented.

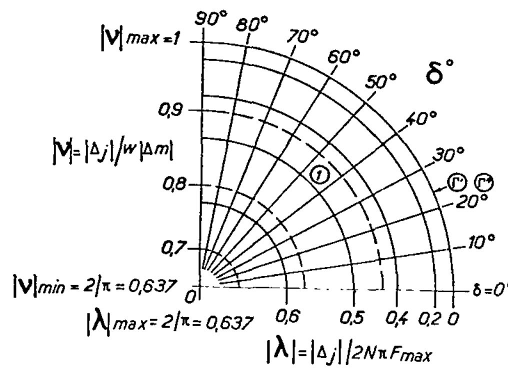


Fig. 14. Nomographic Chart e = 0.

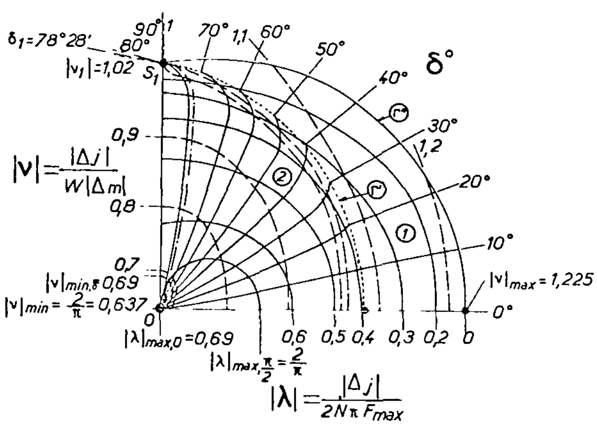


Fig. 15. Nomographic Chart e = 0.2.

Knowing the rotation (δ, Δj) to be made, the number of turns N, the maximal thrust available F_{max}, it is possible to place the point S(δ, $|\lambda|$) on the nomographic chart, which gives $|\nu|$, (therefore the consumption $|\Delta m|$ if we know the ejection velocity W) as well as α and s (namely the configuration of the thrust arcs).

The lines $\delta = C^{te}$ corresponding to values of δ above $\delta_1(e)$ = Arc cos e are situated completely in zone (2). The rotations around axes too distant /84 from the major axis therefore absolutely necessitate two thrust arcs per revolution, no matter what the value of the parameter $|\lambda|$.

This result can be interpreted simply in the case of bi-impulsional solutions corresponding to the point S_1 of the nomographic chart by appealing to the notion of Contensou's "maneuverability domain" [4] (Figure 17):

an impulse dC applied at the point *M*, perpendicular to the plane of the orbit, produces a rotation $\vec{\vec{dj}}$ of the orbit around $\vec{\vec{OM}}$ proportional to O*M*.

Therefore this rotation can be represented by the vector $\vec{\vec{OM}}$, if the impulse is applied upward (z > 0), or the vector $\vec{\vec{O\overline{M}}}$, opposed to $\vec{\vec{OM}}$, if it is applied downward (z < 0).

With the same total impulse dC, it is possible to achieve any rotation /85 the image of which is found inside the smallest *convex* contour surrounding the ellipses which are loci of *M* and $\overline{M}$. As a matter of fact it is enough to breakdown the impulse dC into two or more impulses applied at different points of the orbit.

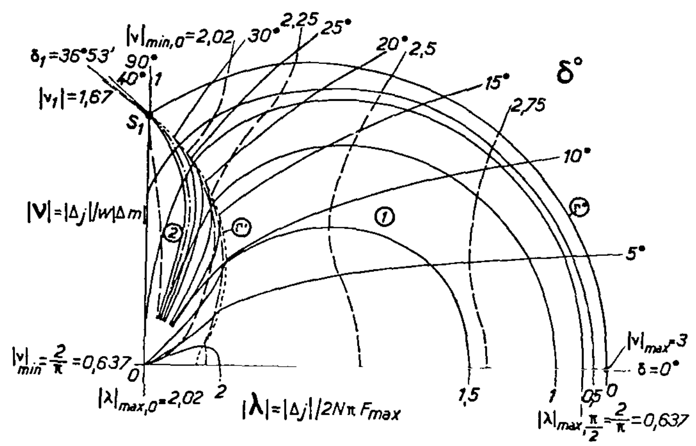


Fig. 16. Nomographic Chart e = 0.8.

The total impulse dC being given, the *maximal* rotations which can be obtained correspond to images on the *contour* of the maneuverability domain thus defined.

If the axis of rotation is far from the major axis, ($\delta > \delta_1$ = Arc cos e), the image μ is found, e.g. on the segment $B\overline{B}'$. Then the rotation $\overrightarrow{O\mu}$ must be broken down into two rotations, generally unequal and respectively around $\overrightarrow{OB}$ and $\overrightarrow{O\overline{B}'}$ found by successively applying at each revolution two unequal impulses directed contrary to the peaks B and B' of the minor axis of the orbit (*O*).

The *impulsional solutions* corresponding to the frontier (Γ'') (mono-impulses) and to the point S_1 (bi-impulses) are such that the parameter $|\lambda|$ is zero. Therefore they are obtained when the maximal thrust available (related to force of attraction at distance a) is very large in respect to the rotation $|\Delta j|$ to be made per revolution.

Given N and F_{max}, a weak rotation, corresponding to a value of $|\lambda|$ such that $0 \leqslant |\lambda| \leqslant |\lambda|_{max,\frac{\pi}{2}} = \frac{2}{\pi}$, can be made around any axis at all.

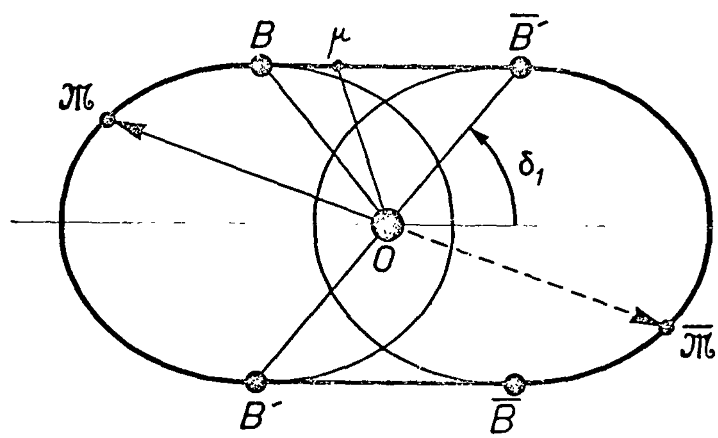


Fig. 17. Maneuverability Domain.

An important rotation, corresponding to $|\lambda|_{max,\frac{\pi}{2}} = \frac{2}{\pi} \leq |\lambda| \leq |\lambda|_{max,o}$,

cannot be made around axes too close to the "parameter" ($\delta < \delta_{lim}(|\lambda|)$ given in Appendix 10.

The maximal obtainable rotation $\left[|\lambda|_{max,o} = \frac{3e}{\sqrt{1-e^2}}\left(\frac{1}{2} - \frac{\delta 1}{\pi}\right) + \frac{2+e^2}{\pi} \right]$ is a rotation around the major axis (δ = 0) obtained by applying thrust continuously.

The maximal specific rotation $|\nu|_{max} = \sqrt{\frac{1+e}{1-e}}$ is obtained in the case of a rotation around the major axis (δ = 0) using an impulse at apogee per revolution ($|\lambda|$ = 0).

Specific rotation diminishes when we go from impulsional solutions to solutions where the thrust is applied continuously. As a matter of fact the thrust arcs are more and more spread out and the thrust is applied in zones where efficiency $|\vec{p}_V|$ is weak.

Minimal specific rotation $|\nu|_{min,\frac{\pi}{2}} = \frac{2}{\pi}$ is obtained for a rotation /86 around the parameter ($\delta = \frac{\pi}{2}$) in the case of continuous thrust. Lines $|\nu| = C^{te}$ corresponding to $1 < |\nu| < |\nu_1| = \frac{1}{\sin \delta_1} = \frac{1}{\sqrt{1-e^2}}$ are entirely situated in zone (2) (two thrust arcs). The values of $|\nu|$ included between 1 and $|\nu_1|$ therefore correspond, among others, to specific rotations relating to bi-impulsional solutions (point S_1).

Particular Cases.

a) *Continuous thrust* (point 0 of the nomographic chart).

for $s = 0,\ \beta' = \beta'' = \beta = \operatorname{Arc\,cos}(e \cos \alpha)$

$$|\lambda_\eta| = \frac{1}{2\pi h}\left[-3e(2\beta - \pi) + 4(1+e^2)\cos\alpha \sin\beta - e\cos 2\alpha \sin 2\beta\right] \quad (27)$$

$$|\lambda_\xi| = \frac{1}{2\pi}\left(4 \sin\alpha \sin\beta - e \sin 2\alpha \sin 2\beta\right) \quad (28)$$

$$|\lambda_c| = 1 \quad (29)$$

whence the values of δ and of $|\nu| = |\lambda|$ given in Appendix 10.

In particular, for $\alpha = 0$, β_0 = Arc cos e = δ_1 (evident in Figure 6) and:

$$|\lambda|_{max,0} = |\nu|_{min,0} = |\lambda_{\eta 0}| = \frac{3e}{\sqrt{1-e^2}}\left(\frac{1}{2} - \frac{\delta_1}{\pi}\right) + \frac{2+e^2}{\pi}. \tag{30}$$

b) *Mono-impulsional and near-mono-impulsional solutions.*

These solutions correspond to points S of the nomographic chart respectively on the frontier (Γ'') and in the vicinity of this frontier.

Making $\beta'' = \pi - \varepsilon$ $(\varepsilon << \pi)$ in equations (24), (25), and (26) we derive:

$$|\nu| = \frac{|\Delta j|}{W/|\Delta m|} = \frac{\sqrt{1-e^2}}{1-e\cos\delta}\left(1 - \frac{\pi^2}{6}\frac{(1-e\cos\delta)^5}{(1-e^2)^4}\lambda^2 + \text{order } |\lambda|^3\right) =$$

$$\frac{\sqrt{1-e^2}}{1-e\cos\delta}\left(1 - \frac{\pi^2}{6}\frac{(1-e\cos\delta)^3}{(1-e^2)^3}\lambda_c^2 + \text{order } \lambda_c^3\right). \tag{31}$$

This formula *directly* furnishes the specific rotation $|\nu|$ as a function of the parameters $|\lambda|$ and δ, that is to say as a function of δ, $|\Delta j|$, N and F_{max}.

Considering the mono-impulsional solution ($\lambda = 0$) of Figure 17, we again find:

$$|\nu|_{imp.} = \frac{|\Delta j|}{\Delta C} = \frac{O\mathcal{M}}{h} = \frac{r}{h} = \frac{\sqrt{1-e^2}}{1-e\cos\delta} \tag{32}$$

maximal for $\delta = 0$:

$$|\nu|_{imp.,max.} = \sqrt{\frac{1+e}{1-e}}. \tag{33}$$

The extension of the thrust arc around the point of impulse causes a relative *loss* equal to /87

$$\frac{\pi^2}{6}\frac{(1-e\cos\delta)^3}{(1-e^2)^3}\lambda_c^2 \quad \text{where} \quad \lambda_c = \frac{\Delta t \text{ propulsion on an arc}}{T(\text{period})}.$$

This loss is *maximal* for $\cos\delta = \cos\delta_1 = e$ and is equal to $\frac{\pi^2}{6}\left(\frac{\Delta t_{prop}}{T}\right)^2$ no matter what e may be, in agreement with the results shown in [26].

c) *Bi-impulsional solutions* (point S_1).

Considering the bi-impulsional solution of Figure 17, we directly obtain:

$$|\nu|_{bi-imp.} = \frac{|\Delta j|}{\Delta C} = \frac{O\mu}{h} = \frac{b}{h \sin\delta} = \frac{1}{\sin\delta}. \tag{34}$$

d) *Elliptical orbit of weak eccentricity* ($\varepsilon << e << 1$).

This is a very important case in practice.

A limited development according to the increasing powers of eccentricity e, clearly stated in Appendix 11, leads to:

Zone (1): *1 thrust arc per revolution*:

$$z = \frac{\pi|\lambda|}{2} \leqslant \sqrt{e \cos\delta} + \text{order } e^{3/2} \tag{35}$$

$$|\nu| = \frac{|\Delta j|}{W/\Delta m|} = 1 + e\cos\delta - \frac{\pi^2 \lambda^2}{6} + \text{order } e^2. \tag{36}$$

Zone (2): *2 thrust arcs per revolution*:

$$z = \frac{\pi|\lambda|}{2} > \sqrt{e \cos\delta} + \text{order } e^{3/2} \tag{37}$$

$$|\nu| = \frac{|\Delta j|}{W|\Delta m|} = \frac{z}{\operatorname{Arc\,sin} z} + \frac{1+3z^2}{2(\operatorname{Arc\,sin} z)^2 \sqrt{1-z^2}} e^2 \cos^2\delta + \text{order } e^4. \tag{38}$$

In the particular case of the circle, expression of the specific rotation is very simple:

$$|\nu|_{e=o} = \frac{\frac{\pi}{2}|\lambda|}{\operatorname{Arc\,sin}\left(\frac{\pi}{2}|\lambda|\right)}. \tag{39}$$

In this case there are always two thrust arcs per revolution and they are symmetrical in relation to zero.

II,2.3.2.3. Comparison between propulsion systems (S_1) and (S_2).

The comparison will only be made for rotation around the major axis $\overrightarrow{Ox}$ ($\delta = 0$) and rotation around the "parameter" $\overrightarrow{Oy}$ ($\delta = \frac{\pi}{2}$).

In Figure 13 we have shown the reduced rotation $|\zeta| = \frac{|\Delta j|}{\sqrt{2}\, P_{max} |\Delta m| \Delta t}$ as a function of e in the two following cases:

a) *Continuous thrust* [Propulsion system (S_{1c})]

$$|\zeta|_{\delta=0} = |\lambda|_{max,0} = \frac{3\,e}{\sqrt{1-e^2}}\left(\frac{1}{2} - \frac{\text{Arc cos } e}{\pi}\right) + \frac{2+e^2}{\pi}$$

$$|\zeta|_{\delta=\frac{\pi}{2}} = |\lambda|_{max,\frac{\pi}{2}} = \frac{2}{\pi} \quad \text{(independent of e).}$$ /88

As in the case of the system (S_2), the greater the eccentricity the less costly is rotation around the major axis in relation to rotation around the "parameter".

For $\delta = \frac{\pi}{2}$, the penalty due to not modulating the ejection velocity is constant, no matter what e is, and equal to the value already obtained in the circular case:

$$\frac{\Delta|\zeta|}{|\zeta|} = \frac{\frac{1}{\sqrt{2}} - \frac{2}{\pi}}{\frac{1}{\sqrt{2}}} \simeq 0.10 \text{ or } 20\% \text{ of mass consumption.}$$

b) *Optimal ejection velocity* W* [System (S_1^*)]

For $\delta = 0\ (\rightarrow \alpha = 0)$ and $\delta = \frac{\pi}{2}\left(\rightarrow \alpha = \frac{\pi}{2}\right)$, $|\lambda|$ and $|\nu|$ are only a function of s. For the choice $W = W^*$, $|\zeta| = \sqrt{\lambda\nu}$ is maximal in relation to s.

Let us first treat the case $\delta = \frac{\pi}{2}$. Then:

$$|\lambda| = \frac{2}{\pi}\sin\beta\ , \quad \lambda_c = \frac{2}{\pi}\beta$$

with β = Arc cos s (independent of e)

therefore:

$$|\zeta^*| = \max_{\beta}\frac{|\lambda|}{\sqrt{\lambda_c}} = \sqrt{\frac{2}{\pi}}\max_{\beta}\frac{\sin\beta}{\sqrt{\beta}} = \sqrt{\frac{2}{\pi}}\,\frac{\sin\beta^*}{\sqrt{\beta^*}}$$

where β^* is the root included between 0 and $\frac{\pi}{2}$ of the transcendant equation:

$$tg\ \beta = 2\beta. \tag{40}$$

An approximate value of β^* is $\beta^* = 66.75°$, which gives $|\zeta^*| \approx 0.68$.

In the case $\delta = 0$, it is necessary to find the point furnishing $|\zeta|_{max}$ corresponding to a value of e on each nomographic chart. The result is shown in Figure 13. The penalty due to not modulating the ejection velocity is weaker than in the case of continuous thrust.

In the circular case, this penalty is equal to:

$$\frac{\Delta|\zeta|}{|\zeta|} = \frac{\frac{1}{\sqrt{2}} - 0,68}{\frac{1}{\sqrt{2}}} = 0,038$$

or only about 8% of mass consumption instead of 20%.

In this circular case the value β^* is easily found by applying condition (I,3 - 80). The area (1) of the cylindrical surface between the commutation level z = +1 and the ellipse (E) must be equal to area (2) (Figure 18). Let it be:

$$\int_0^{\beta^*} \cos L \, dL = 2\beta^* \cos\beta^*$$

or else

$$\sin\beta^* = 2\beta^* \cos\beta^*$$

which is the condition obtained in (40).

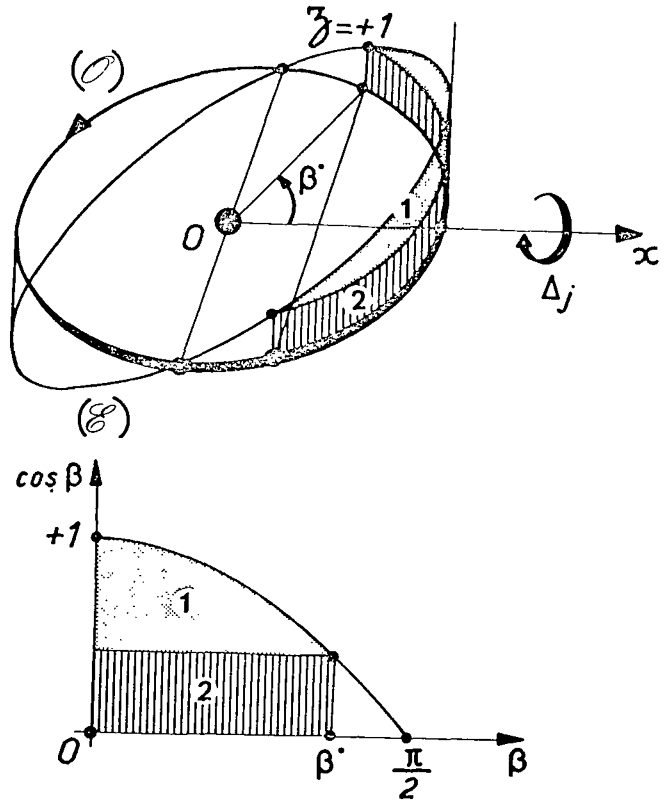


Fig.18. Optimal Length of Maximal Thrust Arcs.

II,2.4. Conclusion. /89

Economic infinitesimal rotation of the plane of an elliptical Keplerian orbit is found by applying the thrust perpendicularly to the plane of the orbit.

For the propulsion system (S_2): W variable, $P = P_{max}$, the thrust is modulated and the thrust extremity of the vector describes an ellipse.

For propulsion system (S_1): $W = C^{te}$, $F \leq F_{max}$ maximal thrust is applied on one or two arcs at each revolution.

For the two systems, the thrust "couple" makes the orbit turn in a direction approximately perpendicular to the axis of the "couple" by a "gyroscopic effect".

In the particular case of a circular orbit and in the case of any orbit for a whole number of revolutions, there is no "uncoupling" respectively between the rotation around the axis of symmetry of the transfer arc and a perpendicular axis, or between rotation around the major axis and rotation around the "parameter", except for the propulsion system (S_2): W variable, $P = P_{max}$.

For the propulsion system (S_1): $W = C^{te}$, $F \leq F_{max}$, there is still mutual non-induction between these rotations.

The penalty in mass consumption due to not modulating ejection velocity W is generally of the order of 20% if the maximal thrust is applied continuously and only of the order of 8% if (constant) ejection velocity W is optimized.

Finally, the previous study has emphasized the fact that rotations around the major axis are less expensive than rotations around the "parameter", especially for large eccentricities.

II,3. OPTIMAL TRANSFERS BETWEEN CLOSE COPLANAR CIRCULAR ORBITS /90

II,3.1. Introduction.

This is a matter of achieving, for a given transfer angle ΔL (Figure 1) the relative, optimal dilatation $\Delta a/a$ of the radius of a circular orbit.

We suppose that there is no rendezvous.

In the case of propulsion system (S_2) with a variable ejection velocity W and limited power, the problem has been analytically resolved by Gobetz [38] without using the orbital elements as state coordinates.

It has been taken up again by the author [45], this time using the orbital elements, then extended by Gobetz under the same conditions to the rendezvous case [39].

In the case of propulsion system (S_1) with a constant ejection velocity W and limited thrust ($F \leq F_{max}$), the problem has been resolved by an analytico-numerical method by McIntyre and Crocco [41-44] and by Hinz [52] (the latter treated only the case of continuous constant thrust), without using the orbital elements.

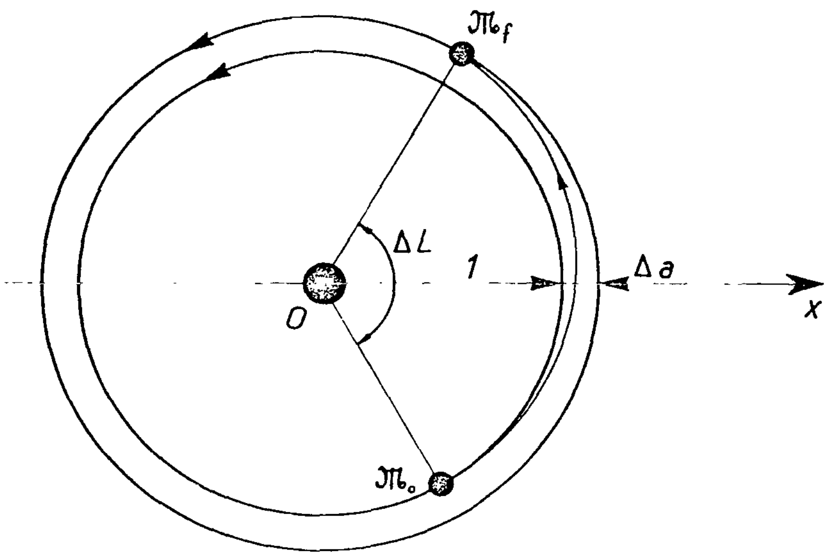


Fig. 1. Transfer Between Close Circles.

The introduction of the orbital elements into the present study brings with it a few simplifications and particularly permits us to obtain an approximate explicit formula giving consumption in most cases [45].

The only components of the kinematic adjoint P which are not necessarily zero are: p_a, p_α and p_β.

We shall posit: /91

$$\begin{cases} p_\alpha = p_e \cos L_e \\ p_\beta = p_e \sin L_e \end{cases} \tag{1}$$

The origin of the arcs will be taken in the middle of the transfer arc $\widehat{M_0 M_f}$:

$$\begin{cases} L_o = -\dfrac{\Delta L}{2} \\ L_f = +\dfrac{\Delta L}{2} \end{cases} \tag{2}$$

II,3.2. Optimal Thrust.

II,3.2.1. Efficiency curve.

In § I,3.2.4. we have already seen that the efficiency curve (P) is an *ellipse* (Figure 2):

$$\vec{P_v} \begin{cases} X = p_e \sin(L - L_e) \\ Y = 2p_a + 2p_e \cos(L - L_e) \end{cases} \tag{3}$$

deduced from the circle (C) with center ω ($X = 0$, $Y = 2p_a$) and with radius $2p_e$ through orthogonal affinity of axis $\overrightarrow{MY}$ and of ratio 1/2.

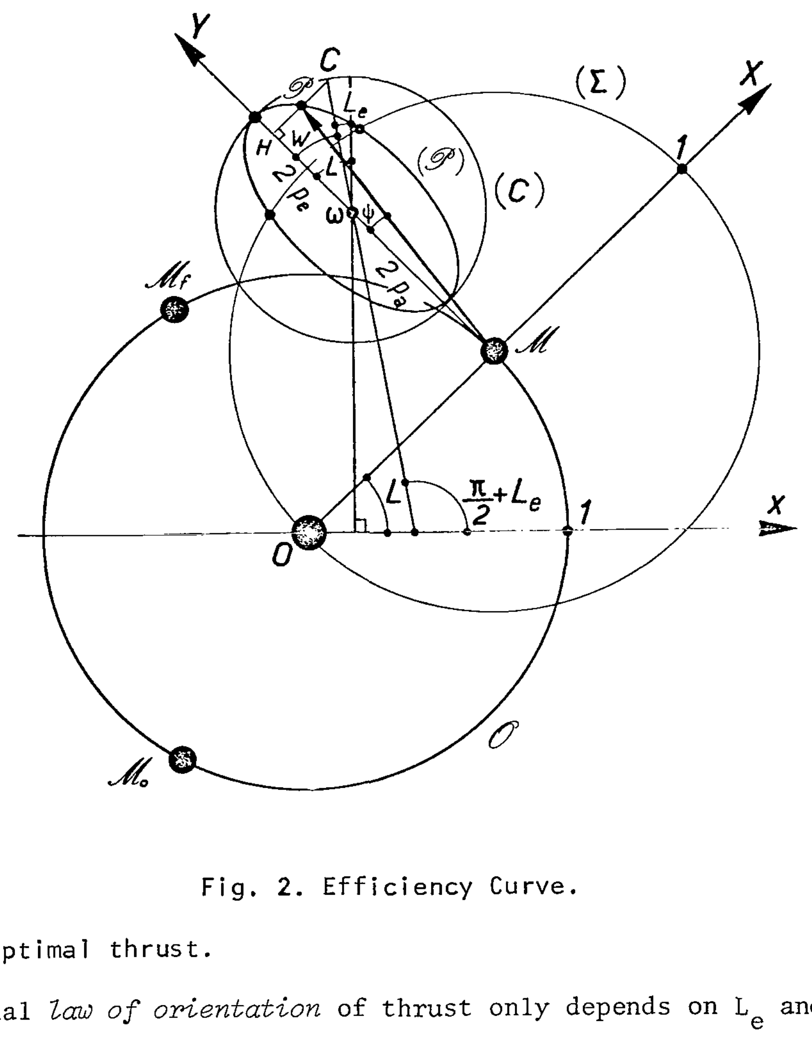


Fig. 2. Efficiency Curve.

II,3.2.2. Optimal thrust. /92

The optimal *law of orientation* of thrust only depends on L_e and on ratio:

$$q = \frac{p_a}{p_e} \gtrless 0 \tag{4}$$

i.e. on L_e and on the position of *M* in relation to *(P)*.

It is given by:

$$tg\,\psi^* = \frac{X}{Y} = \frac{\sin(L-L_e)}{2\,[q+\cos(L-L_e)]} \tag{5}$$

In order to follow easily the variations of the optimal orientation of thrust, it is possible to use a construction different from that in Figure 2, although strictly equivalent, by noticing that $w = L - L_e$ is, in Figure 2, nothing but the eccentric anomaly of point P:

Let $\overrightarrow{Ox_1}$ be the axis deduced from $\overrightarrow{Ox}$ by rotation $+ L_e$ (Figure 3).

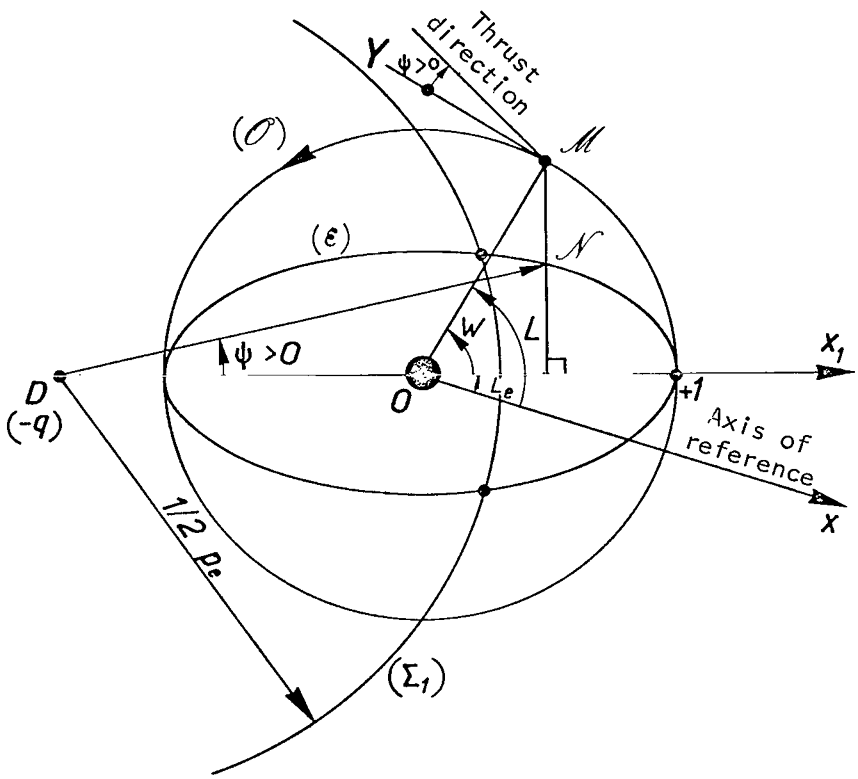


Fig. 3. Optimal Thrust Direction.

The ellipse (E) closely related to the circular orbit (O) is plotted in the orthogonal affinity of axis $\overrightarrow{Ox_1}$ and of ratio 1/2.

The point D is placed on Ox_1 so that $\overline{OD} = -q \lesssim 0$. The angle ψ is then the angle $(\overrightarrow{Dx_1}, \overrightarrow{DM})$. Thus the thrust direction is deduced directly from the position of point M on the orbit (O).

Let us note that during a revolution ψ sweeps an angle $\leq \pi$ if $|q| \geq 1$ and an angle 2π if $|q| < 1$.

ψ undergoes a *discontinuity* equal to π for $q = +1$, $w = \pi$ or $q = -1$, $w = 0$.

If we change L_e into $L_e + \pi$ and q into -q (that is to say p_a into $-p_a$) /93 *the position of* D *does not change*, but ψ is changed into $\psi + \pi$, i.e. thrust is *reversed*.

It is evident that then Δa is changed into $-\Delta a$.

OPTIMAL THRUST MODULUS.

1. *Propulsion system* (S_2): W variable, $p = p_{max}$.

The optimal modulus of thrust acceleration is:

$$\gamma = |\overrightarrow{\mathcal{M}\mathcal{P}}| = \left[p_e^2 \sin^2 w + 4\left(p_a + p_e \cos w\right)^2\right]^{\frac{1}{2}} = \frac{D\mathcal{N}}{2 p_e}. \tag{6}$$

The evolution of γ, when L (therefore w) varies, can be directly followed on Figure 3.

2. *Propulsion system* (S_1): $W = C^{te}$, $F \leq F_{max}$.

$$F = |\vec{F}| = \begin{cases} F_{max} \\ 0 \end{cases} \quad \text{according as} \quad |\vec{p_v}| = |\overrightarrow{\mathcal{M}\mathcal{P}}| \gtrless 1 \tag{7}$$

i.e. P on $\left\{\begin{matrix}\text{the exterior}\\ \text{the interior}\end{matrix}\right\}$ of the circle (Σ) centered at M and of radius 1, (Figure 2) or else N on $\left\{\begin{matrix}\text{the exterior}\\ \text{the interior}\end{matrix}\right\}$ of the circle (Σ_1) of center D and of radius 1/2 p_e (Figure 3).

When $p_e \to 0$, the efficiency curve is reduced to point $\omega(\overline{M\omega} = 2p_a)$. When the thrust is applied it is horizontal. According to whether the point is interior to $(\Sigma)(|p_a| < 1/2)$, exterior to $(\Sigma)(|p_a| > 1/2)$ or on $(\Sigma)(|p_a| = 1/2)$ there is an absence of thrust, continuous thrust or *a singular solution* $(|\vec{\vec{p}}_v| \equiv 1)$ of the type I bis (see § I,3.2.6. and II,1.2.).

II,3.2.3. Directrix Curve (D).

If the normalization condition of the adjoint is such that $|\vec{\vec{p}}_v| \ll 1$, the locus of P in absolute space is a Keplerian ellipse (O_P) of focus 0, the elements of which are given by (I,3.48-49):

$$\begin{cases} \overrightarrow{e_{\mathscr{P}}} = \vec{e} + \overrightarrow{De} = \vec{p_e} \wedge \vec{z} \\ a_{\mathscr{P}} = a \end{cases} \tag{8}$$

Figure 4 has been plotted in the case where $L_e = \pi$.

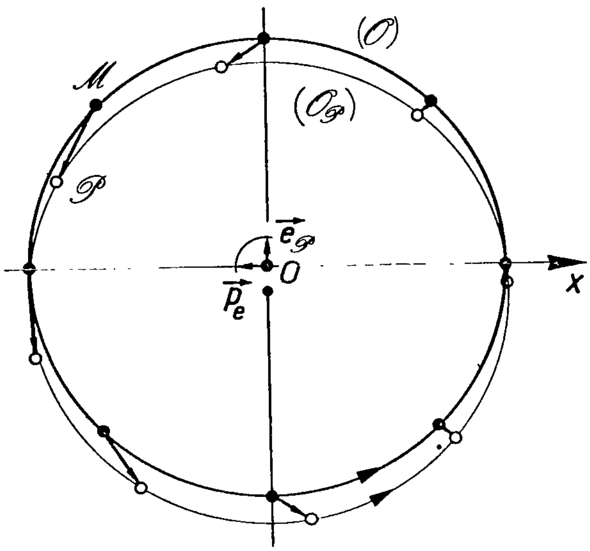


Fig. 4. Directrix Orbit.

II,3.3. Optimal "Dilatation" of the Radius of the Circular Orbit. Consumption.

II,3.3.1. Propulsion System (S_2).

Equation (I,3 - 70) furnishes the adjoint as a function of the variations imposed: /94

$$P^T = \begin{vmatrix} p_a \\ p_\alpha \\ p_\beta \end{vmatrix} = \langle G \rangle^{-1} \frac{\Delta \mathscr{X}}{\Delta L} =$$

$$\begin{vmatrix} \langle G_{aa} \rangle & \langle G_{a\alpha} \rangle & 0 \\ \langle G_{a\alpha} \rangle & \langle G_{\alpha\alpha} \rangle & 0 \\ 0 & 0 & \langle G_{\beta\beta} \rangle \end{vmatrix}^{-1} \begin{vmatrix} \frac{\Delta a}{\Delta L} \\ \frac{\Delta \alpha}{\Delta L} \\ \frac{\Delta \beta}{\Delta L} \end{vmatrix} = \begin{vmatrix} \frac{\langle G_{\alpha\alpha} \rangle}{\delta} & -\frac{\langle G_{a\alpha} \rangle}{\delta} & 0 \\ -\frac{\langle G_{a\alpha} \rangle}{\delta} & \frac{\langle G_{aa} \rangle}{\delta} & 0 \\ 0 & 0 & \frac{1}{\langle G_{\beta\beta} \rangle} \end{vmatrix} \begin{vmatrix} \frac{\Delta a}{\Delta L} \\ 0 \\ 0 \end{vmatrix}$$

with:

$$\delta = \langle G_{aa} \rangle \langle G_{\alpha\alpha} \rangle - \langle G_{a\alpha} \rangle^2 \geqslant 0$$

whence:

$$\begin{cases} p_a = \dfrac{\langle G_{\alpha\alpha} \rangle}{8} \dfrac{\Delta a}{\Delta L} & (\langle G_{\alpha\alpha} \rangle > 0) \\ p_\alpha = -\dfrac{\langle G_{a\alpha} \rangle}{8} \dfrac{\Delta a}{\Delta L} \; (= \varepsilon_e \, p_e) \\ p_\beta = 0 \Longrightarrow L_e = \begin{cases} 0 \\ \pi \end{cases} \text{ or } \; \varepsilon_e = \cos L_e = \pm 1 \end{cases} \tag{9}$$

and in particular,

$$q = \frac{p_a}{p_e} = -\varepsilon_e \frac{\langle G_{\alpha\alpha} \rangle}{\langle G_{a\alpha} \rangle} = -\varepsilon_e \frac{5 + 3 \dfrac{\sin \Delta L}{\Delta L}}{8 \dfrac{\sin(\Delta L/2)}{\Delta L/2}} \tag{10}$$

It is sufficient to study the cases where $L_e = 0$ and $\varepsilon_e = +1$. Then:

$\Delta a \gtrless 0$ according to $\langle G_{a\alpha} \rangle \lessgtr 0$ or:

$$\begin{cases} (4k+2)\pi < \Delta L < 4(k+1)\pi \\ 4k\pi < \Delta L < (4k+2)\pi \end{cases}$$

Change p_α into $-p_\alpha$, that is to say ε_e into $-\varepsilon_e$, change q into -q (see 10), /95 that is to say p_a into $-p_a$. Thrust is reversed and Δa is changed into $-\Delta a$.

If therefore, , for the given angle of transfer, ΔL, the *sign* of Δa obtained with the hypothesis $\varepsilon_e = +1$ is not suitable, *it is sufficient to reverse the thrust.*

Equation (10) shows that *the optimal law of orientation of thrust is only a function of the transfer angle.*

When ΔL increases from $\Delta L = 0$, q begins from value q = -1 and remains slightly superior to -1 for $0 < \Delta L < \overline{\Delta L} \simeq \pi$ (Figure 5). The value $\overline{\Delta L}$ is the root > 0, not zero, of the transcendant equation:

$$5 + 3 \frac{\sin \Delta L}{\Delta L} = 8 \frac{\sin(\Delta L/2)}{\Delta L/2} . \tag{11}$$

An approximate root is: $\overline{\Delta L} = 187°$.

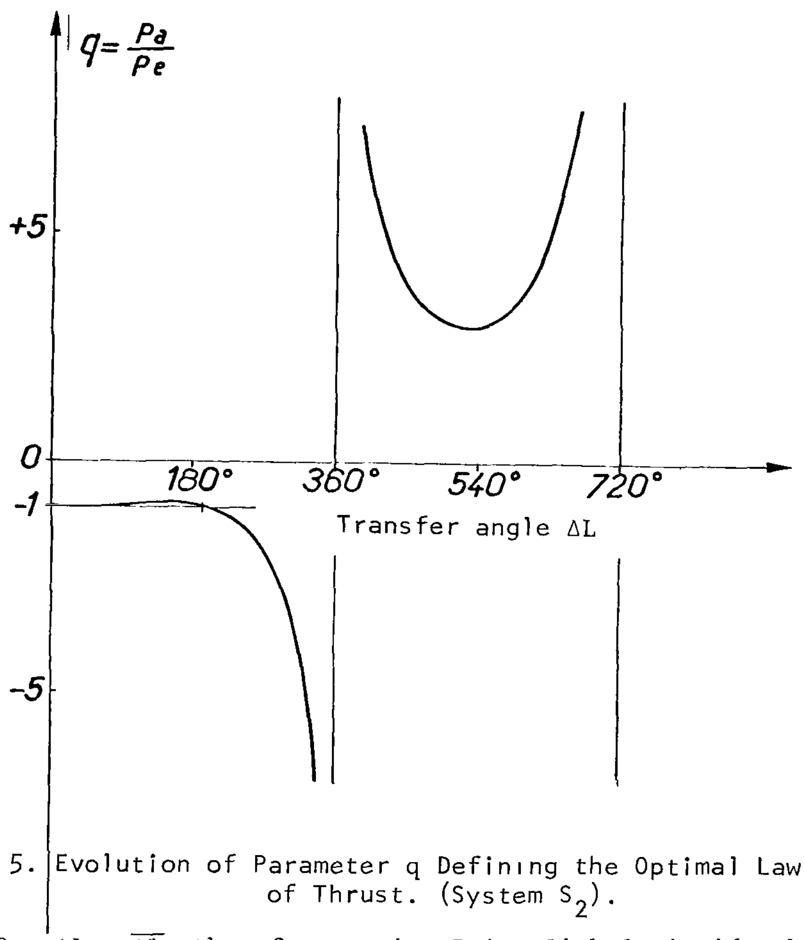


Fig. 5. Evolution of Parameter q Defining the Optimal Law of Orientation of Thrust. (System S_2).

For $0 < \Delta L < \overline{\Delta L}$, therefore, point D is slightly inside the ellipse (*E*) (Figure 6). The corresponding optimal thrust is plotted in this figure. *Inversion* of the thrust has been carried out so that $\Delta a > 0$ (*). Again we find the acceptance of the optimal thrust laws found by numerical methods, as in [34] and [35]. It is important to note the reversal of the tangential thrust in the central zone of the arc $\widehat{M_0 M_f}$. /96

The extension 2β of the reversal zone for $\Delta L \ll \pi$ is:

$$2\beta \sim \frac{\Delta L}{\sqrt{6}}. \tag{12}$$

When $\Delta L > \overline{\Delta L}$, $|q|$ is > 1. The point is outside the ellipse (*E*)(Figure 7). There is no longer a reversal zone of tangential thrust.

(*) The result being that $\psi = (\vec{DO}, \vec{DN})$ and not $(\vec{Dx}, \vec{DN})$.

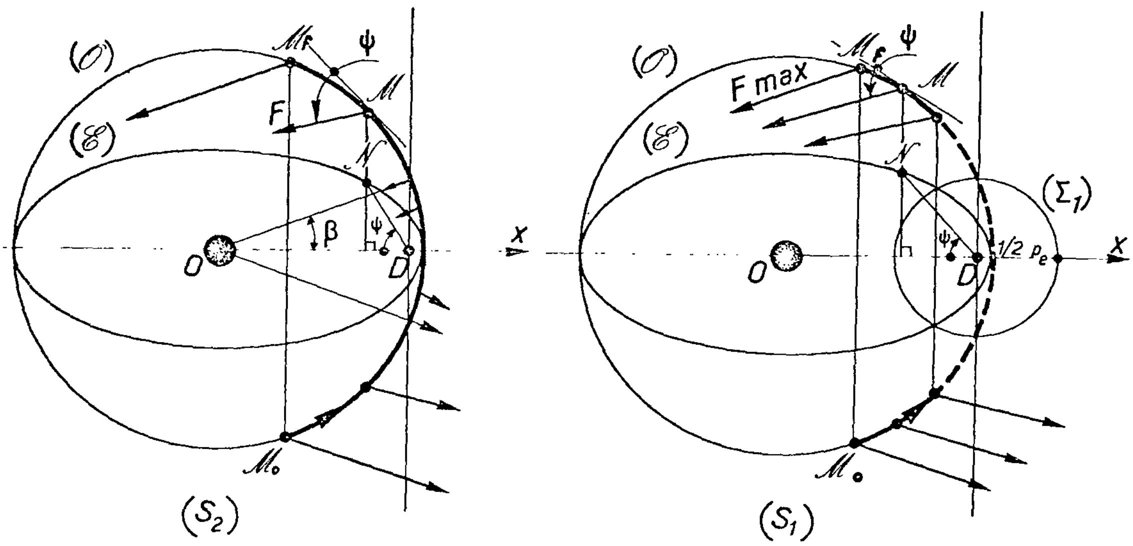


Fig. 6. Optimal Thrust Law. (Weak Transfer Angle).

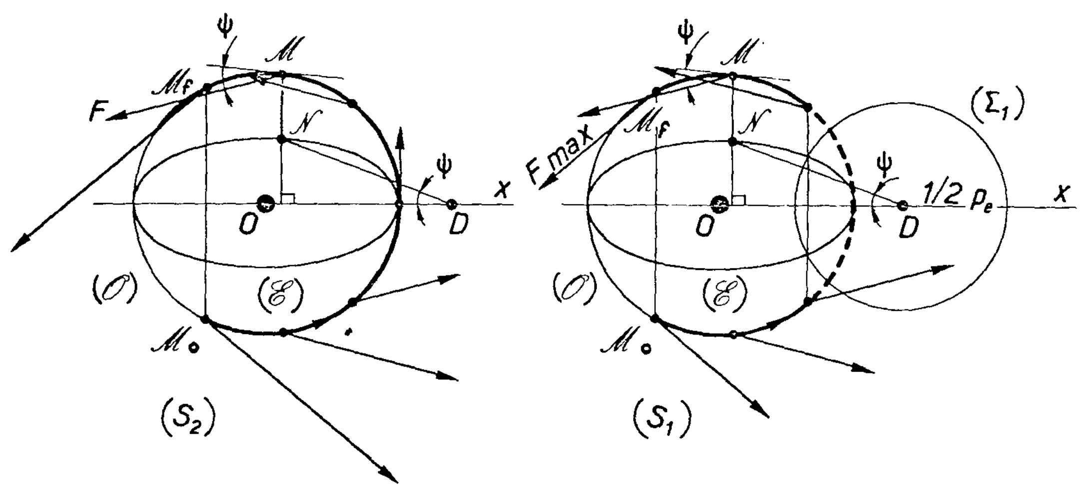


Fig. 7. Optimal Thrust Law. (Significant Transfer Angle).

When $\Delta L = 2N\pi$ (or $\Delta L >> 2\pi$), $p_e = 0$. The efficiency curve is reduced to /97
a point ω. Angle ψ is always zero. Thrust is constant and tangential.

CONSUMPTION.

The reduced dilatation is given by equation (I,3 - 73):

$$|\zeta| = \frac{|\Delta a|}{\sqrt{2P_{max}}\,|\Delta m|\,\Delta L} = \left(\frac{\delta}{<G_{\alpha\alpha}>}\right)^{1/2} = 2\left[1 - \frac{8\left(\frac{\sin\frac{\Delta L}{2}}{\frac{\Delta L}{2}}\right)^2}{5 + 3\,\frac{\sin\Delta L}{\Delta L}}\right]^{1/2} \quad (13)$$

In Figure 8 have been plotted the variation of this "reduced dilatation" $|\zeta|$ as a function of the transfer angle ΔL. When ΔL increases from $\Delta L = 0$, the specific dilatation abandons the value zero and after a few oscillations stabilizes at the value 2, a value which it reaches without passing it, for $\Delta L = 2N\pi$ (whole number of revolutions).

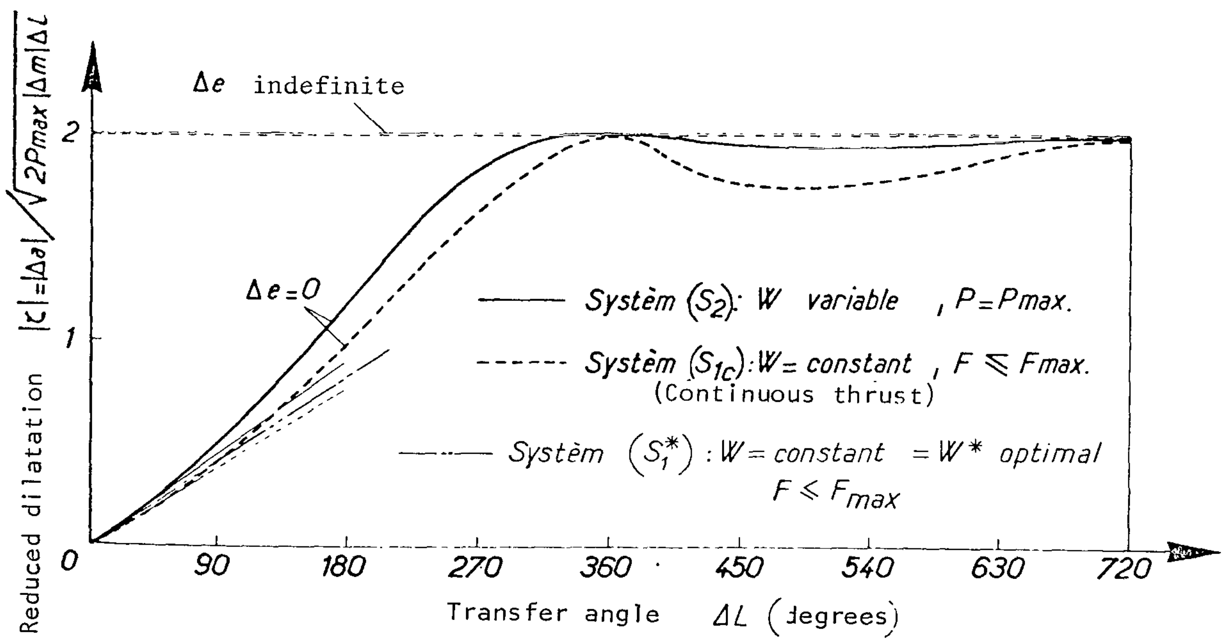


Fig. 8. Comparison of Performance of Propulsion Systems.

If instead of imposing $\Delta e = 0$, the variation of e is indefinite, the "reduced dilatation" is:

$$|\zeta| = \frac{|\Delta a|}{\sqrt{2P_{max}}\,|\Delta m|\,\Delta L} = \sqrt{<G_{aa}>} = 2 \text{ (independent of } \Delta L)$$

Thrust is tangential and constant. /98

Figure 8 clearly shows the penalty which condition $\Delta e = 0$ entails, particularly when the transfer angle is weak. The penalty is zero for $\Delta L = \begin{cases} 2N\pi \\ \infty \end{cases}$.

As a matter of fact, we have seen in §I,4.1.2. that there is then uncoupling between the variations of a and e.

CASE WHERE THE TRANSFER ANGLE IS WEAK ($\Delta L \simeq 0$).

The principal part of $|\zeta|$, calculated from (13) is:

$$|\zeta| \sim \frac{\Delta L}{2\sqrt{3}} \tag{14}$$

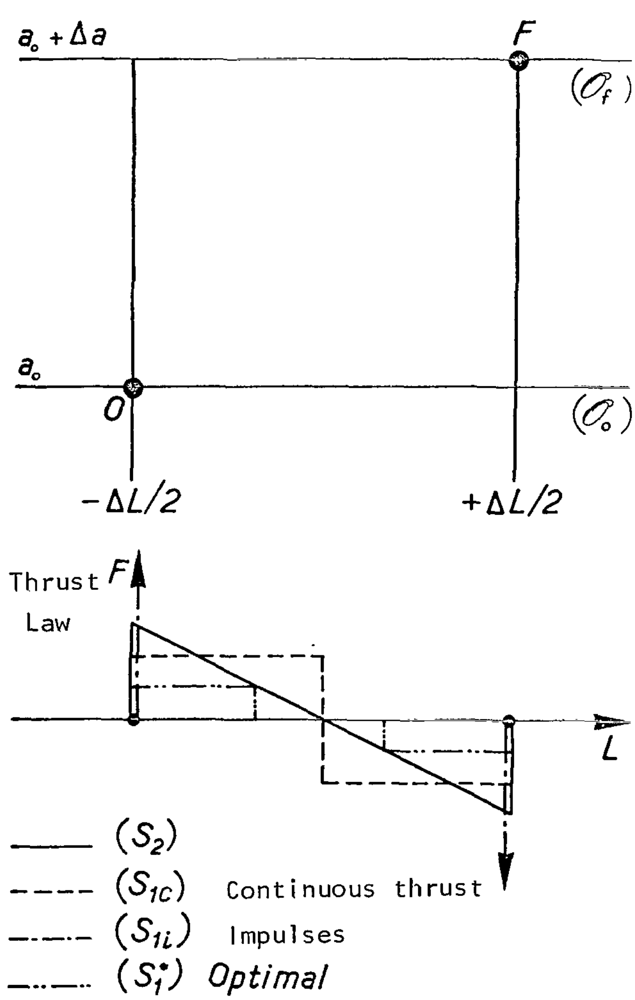


Fig. 9. Very Small Transfer Angle.

which furnishes the tangent at the origin of the curve traced in Figure 8.

This result can be found again very simply: when the transfer angle is very weak, the thrust acceleration necessary to achieve the transfer is very large compared with the gravity acceleration. Everything happens as if we were operating in the absence of gravitational forces. It is then easy to demonstrate that, in order to travel the length $|\Delta a|$ in the time ΔL with a zero velocity at departure and arrival, the most economical method is to use a *linear* law of acceleration variation in L (Figure 9).

Cost is then:

$$\Delta J = \frac{6 \Delta a^2}{\Delta L^3} \tag{15}$$

whence the result (14).

II,3.3.2. Propulsion System (S_1): $W = C^{te}$, $F \leq F_{max}$. /99

II,3.3.2.1. Generalities.

The most rapid method is a direct integration of the equations (I,3 - 65):

$$\Delta a = \int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{|\vec{p_v}|} (XK_{aX} + YK_{aY})\, dL \tag{16}$$

$$\Delta \alpha = 0 = \int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{|\vec{p_v}|} (XK_{\alpha X} + YK_{\alpha Y})\, dL \tag{17}$$

$$\Delta \beta = 0 = \int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{|\vec{p_v}|} (XK_{\beta X} + YK_{\beta Y})\, dL \tag{18}$$

with:

$$|\vec{p_v}| = \left(3p_e^2 \cos^2 w + 8\, p_a\, p_e \cos w + 4p_a^2 + p_e^2\right)^{1/2} \tag{19}$$

$$F = F_{max}\, U(\Theta) \qquad \left(\Theta = |\vec{p_v}| - 1\, ;\; U(\Theta) = \frac{1 + \mathrm{sign}\ \Theta}{2}\right) \tag{20}$$

$$XK_{aX} + YK_{aY} = 4\,(p_a + p_e \cos w) \tag{21}$$

$$\begin{aligned} XK_{\alpha X} + YK_{\alpha Y} &= p_e \sin w \sin(w + L_e) + 4p_a \cos(w + L_e) + 4p_e \cos w \cos(w + L_e) = \\ &(3p_e \cos^2 w + 4p_a \cos w + p_e)\cos L_e - \sin w\,(3p_e \cos w + 4p_a)\sin L_e \end{aligned} \tag{22}$$

$$\begin{aligned} XK_{\beta X} + YK_{\beta Y} &= -p_e \sin w \cos(w + L_e) + 4p_a \sin(w + L_e) + 4p_e \cos w \sin(w + L_e) = \\ &(3p_e \cos^2 w + 4p_a \cos w + p_e)\sin L_e + \sin w\,(3p_e \cos w + 4p_a)\cos L_e . \end{aligned} \tag{23}$$

The result being:

$$\frac{\Delta a}{F_{max}} = A . \tag{24}$$

$$\frac{\Delta \alpha}{F_{max}} = 0 = B \cos L_e - C \sin L_e \tag{25}$$

$$\frac{\Delta \beta}{F_{max}} = 0 = B \sin L_e + C \cos L_e \tag{26}$$

with:

$$A = \int_{-\frac{\Delta L}{2} - L_e}^{+\frac{\Delta L}{2} - L_e} \frac{F}{F_{max}} \frac{4(p_a + p_e \cos w)}{|\vec{p_v}|} dw \tag{27}$$

$$B = \int_{-\frac{\Delta L}{2} - L_e}^{+\frac{\Delta L}{2} - L_e} \frac{F}{F_{max}} \frac{3 p_e \cos^2 w + 4 p_a \cos w + p_e}{|\vec{p_v}|} dw \tag{28}$$ /100

$$C = \int_{-\frac{\Delta L}{2} - L_e}^{+\frac{\Delta L}{2} - L_e} \frac{F}{F_{max}} \frac{3 p_e \cos w + 4 p_a}{|\vec{p_v}|} dw \; . \tag{29}$$

Equations (25) and (26) are only verified simultaneously if:

$$B = C = 0. \tag{30}$$

Now,

$$C = -\frac{1}{p_e} \int_{|\vec{p_v}|_0}^{|\vec{p_v}|_f} \frac{F(|\vec{p_v}|)}{F_{max}} \, d(|\vec{p_v}|). \tag{31}$$

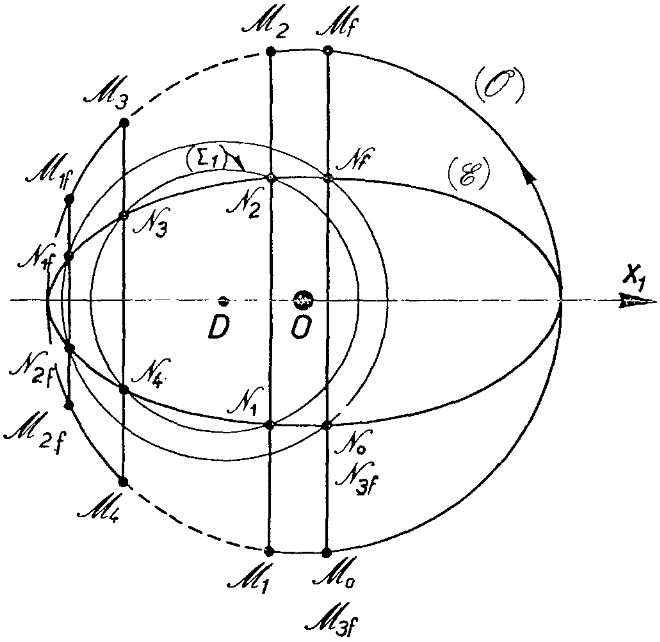


Fig. 10. Determination of L_e.

At the moment we are only considering regular solutions.

The condition C = 0 can only be realized in the following cases (Figure 10):

1. If at least one of the points M_0 or M_f is found on a propulsed arc (let us suppose that it is M_0), it is necessary for $|\vec{p_v}|_f = |\vec{p_v}|_o$, therefore that N_f occupies one of the positions N_f, N_{1f}, N_{2f}, N_{3f}, that is to say that M_f occupies one of the positions M_f, M_{1f}, M_{2f}, M_{3f} situated on a propulsed arc.

The position $M_{3f} = M_0$ signifies that the mobile makes a whole number of revolutions.

This then is equivalent to using the "antisymmetrical" solution in relation to $\overrightarrow{Ox}_1$ (different from $\overrightarrow{Ox}$) or any solution deduced from this by rotation around 0. In particular, we can choose the "antisymmetrical" solution in relation to $\overrightarrow{Ox}$. We shall show that then the *regular* solution is a solution with a continuous tangent thrust ($|q| \to \infty$), and therefore in fact "antisymmetrical" in relation to every axis passing through 0. Therefore there is really/101 no degeneracy by "rotation" of the solution, but rather a unique solution.

Positions M_{1f} and M_{2f} would correspond to solutions where the optimal thrust would be antisymmetrical in relation to the axis $\overrightarrow{Ox}_1$ different from $\overrightarrow{Ox}$, without there being a whole number of revolutions. Such solutions could only exist if D was placed between the two curvature centers of the ellipse (E) situated on the major axis $\overrightarrow{Ox}_1$, therefore for $|q| < 3/4$.

Now, for $|q| < \sqrt{3}/2$, the integrant of B (formula 28) is ≥ 0, therefore B is > 0, since there is at least one propulsed arc.

Therefore positions M_{1f} and M_{2f} are to be rejected.

2. We have just seen that conditions $B = 0$ implies $|q| > \sqrt{3}/2 > 3/4$, therefore that the optimal solutions do not entail any propulsed part of the type $\overset{\frown}{M_3M_4}$.

If M_0M_f are then found on the unique ballistic section $\overset{\frown}{M_2M_1}$, the integrals B and C are not modified when M_0 and M_f coincide respectively with M_1 and M_2. C is really zero then, but B is $\neq 0$.

As a matter of fact we can consider this solution as a limit of a solution where M_0 and M_f, symmetrical in relation to $\overrightarrow{Ox}_1$ and situated on the *propulsed* section $\overset{\frown}{M_1M_2}$, tend respectively toward M_1 and M_2. Now,we shall see that the only solutions of this type which assure that $B = 0$ are such that the arc $\overset{\frown}{M_0M_f} < 2\pi$ contains the ballistic portion $\overset{\frown}{M_2M_1}$. (Here it is really a matter of the arc $\overset{\frown}{M_0M_f} < 2\pi$ and not of the transfer arc $\overset{\frown}{M_0M_f}$ which itself can be $> 2\pi$).

In conclusion, *the regular solutions are always antisymmetrical in relation to* $\overrightarrow{Ox}$($L_e = \{\begin{smallmatrix}0\\ \pi\end{smallmatrix})$ and the points of departure and arrival M_0 and M_f are always the extremities of propulsed arcs but do not generally coincide with the points M_1 and M_2.

Here again it is easy to see that it is possible to *limit oneself to the case* $L_e = 0$. If the sign of Δa found is not convenient, it is enough *to reverse the thrust*.

Equation (26) is verified in an identical manner. The equations (24) and (25) are written:

$$\frac{\Delta a}{F_{max}} = A \equiv \int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{F_{max}} \frac{4(q + \cos L)}{\sqrt{3\cos^2 L + 8q\cos L + 4q^2 + 1}} dL \qquad (32)$$

$$0 = B \equiv \int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{F_{max}} \frac{3\cos^2 L + 4q\cos L + 1}{\sqrt{3\cos^2 L + 8q\cos L + 4q^2 + 1}} dL. \qquad (33)$$

The integrals taking part in the second members are elliptical integrals which could be reduced to the sums of elliptical integrals of the first, second and third species. But integration (for any q) is not absolutely necessary to discuss optimal solutions qualitatively.

II,3.3.2.2. Optimal solutions.

a) Let us consider the function f(x) of the variable x = cos L:

$$f(x) = \frac{B(x)}{4} = -\frac{1}{2}\int_1^x \frac{3x^2 + 4qx + 1}{\sqrt{(1-x^2)(3x^2+8qx+4q^2+1)}} dx. \qquad (34)$$

Let us immediately note that:

$$f_{-q}(x) = f_q(-1) - f_q(-x) \qquad (35)$$ /102

which permits the case $q \geq 0$ to be deduced easily from the study of the case $q \leq 0$.

The function f(x) can also be calculated in an *explicit* fashion for certain values of q:

PARTICULAR CASES (Figure 11).

1/ $\underline{q = 0}$

$$f(x) = -\frac{1}{2}\int_1^x \frac{\sqrt{3x^2+1}}{\sqrt{1-x^2}} dx = \int_0^{L = \text{Arc}\cos x} \sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2 \sin^2 L}\; dL$$

$$f(x) = \left[E\left(k = \frac{\sqrt{3}}{2}, L\right) \right]_0^{L = \mathrm{Arc}\cos x} = E\left(k = \frac{\sqrt{3}}{2}, \; L = \mathrm{Arc}\cos x\right) \tag{36}$$

where F designates the elliptical integral of the second species of modulus $k = \sqrt{3}/2$ (α = Arc sin k = 60°) and of amplitude L = Arc cos x.

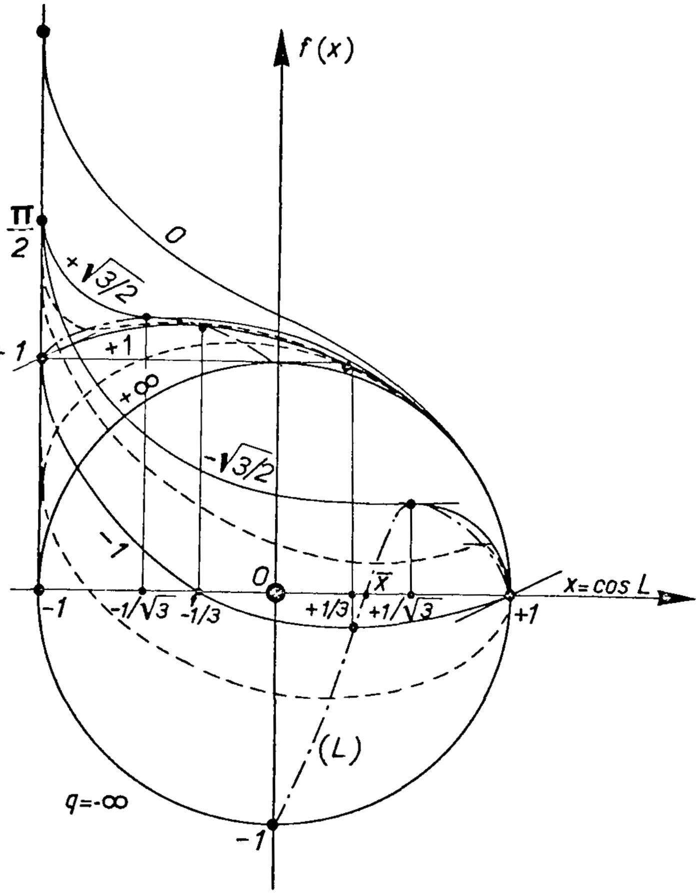


Fig. 11. Function f(x).

$q = -\sqrt{3}/2$ /103

$$f(x) = \frac{1}{2}\int_0^{L=\mathrm{Arc}\cos x} \frac{(1-\sqrt{3}\cos L)^2}{2-\sqrt{3}\cos L}\,dL = \frac{1}{2}\int_0^{L=\mathrm{Arc}\cos x}\left(-\sqrt{3}\cos L + \frac{1}{2-\sqrt{3}\cos L}\right)dL$$

$$f(x) = -\frac{\sqrt{3}}{2}\sin L + \mathrm{Arc}\,tg\,\frac{tg(L/2)}{2-\sqrt{3}} \quad (L = \mathrm{Arc}\cos x). \tag{37}$$

3/ $q = -1$

$$f(x) = -\frac{1}{2}\int_1^x \frac{1-3x}{\sqrt{(1+x)(5-3x)}}\,dx.$$

We posit:

$$y^2 = \frac{1+x}{5-3x} \tag{38}$$

$$f(x) = -4\int_1^y \frac{(1-3y^2)dy}{(3y^2+1)^2} = 1 - \frac{4y}{1+3y^2}. \tag{39}$$

4/ $q = -\infty$

$$f(x) = -\frac{1}{2}\int_0^{L=\mathrm{Arc}\cos x} 2\cos L\,dL = -\sin L = -\sqrt{1-x^2}. \tag{40}$$

b) *For* $0 \leq \Delta L \leq 2\pi$, the optimal solutions correspond to *negative* values of q and to $\Delta a < 0$. There are two thrust arcs, as Figure 12 shows. The point D can be either inside or outside the ellipse (E) (Figure 6 and 7). Points M_0 /104
and M_1 are such that condition (33) is satisfied, namely:

$$f(x_0) = f(x_1). \tag{41}$$

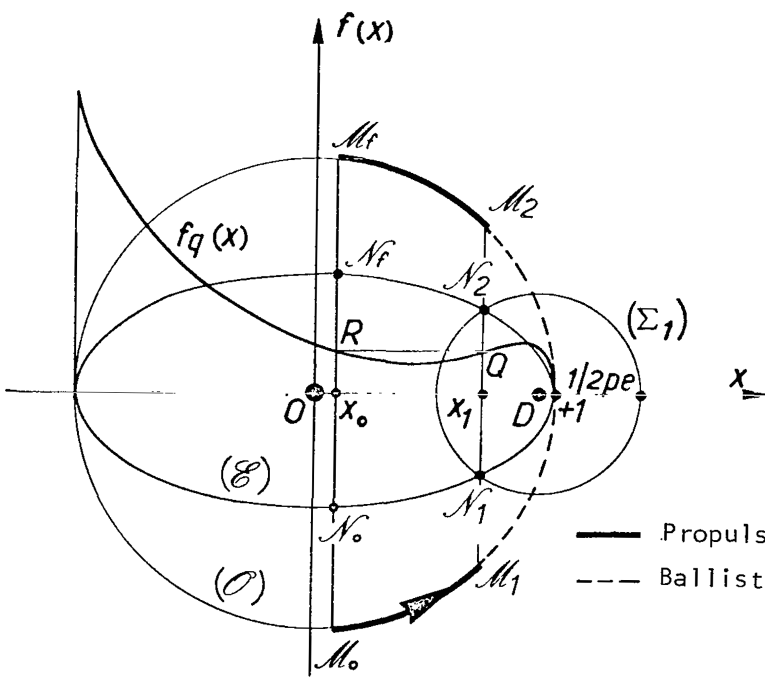


Fig. 12. Optimal Thrust (System S_1). Transfer Angle Less Than One Turn.

Therefore the values x_0 and $x_1 > x_0$ are the abscissae of the intersection points of a particular curve $f = f_q(x)$ with a parallel to axis $\overrightarrow{Ox}$.

The solutions corresponding to a particular value of q or to a particular value of ΔL are studied in detail in Appendix 12.

c) *For* $2\pi \leq \Delta L \leq 4\pi$, the optimal solutions correspond to values of q above 1. There are three thrust arcs, as Figure 13 shows. These are the arcs $\widehat{M_0 M_1}$, $\widehat{M_2 M_1}$, and $\widehat{M_2 M_f}$. The points M_0 and M_1 are such that condition (33) is satisfied, namely:

$$f(x_1) - f(x_0) + f(x_1) - f(0) = 2f(x_1) - f(x_0) = 0 \,. \tag{42}$$

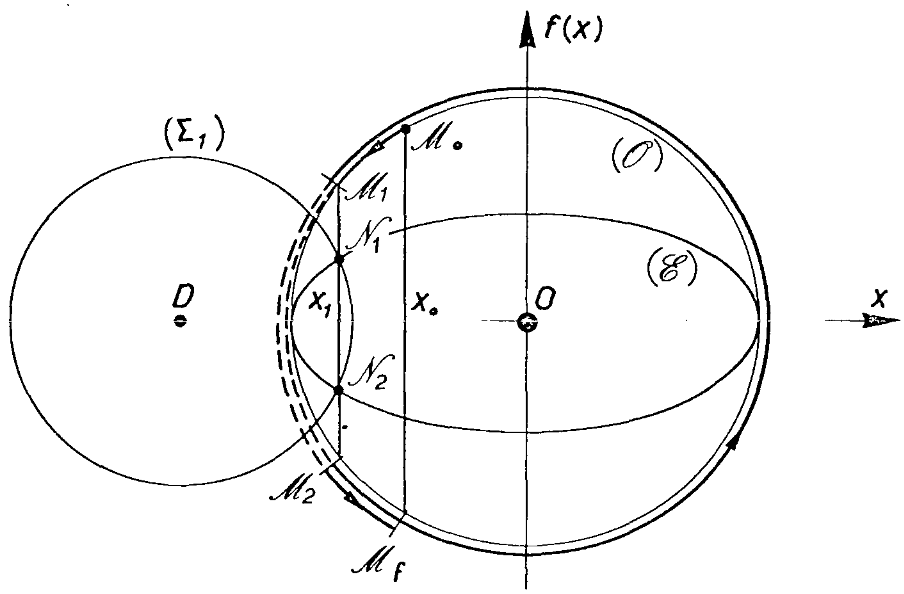


Fig. 13. Optimal Thrust (System S_1). Transfer Angle Between One and Two Turns.

Therefore the values x_0 and $x_1 < x_0$ are the abscissae of the intersection points of a parallel to axis $\overrightarrow{Ox}$ with, respectively, a curve $f = f_q(x)$ and its transform 2f by affinity to axis $\overrightarrow{Ox}$, in direction $\overrightarrow{Of}$ and of ratio 2.

These solutions are likewise discussed in Appendix 12.

d) *For* $2N\pi \leq \Delta L \leq 2(N + 1)\pi$, q is $\gtrless 0$ according to whether

N is $\left\{\begin{matrix}\text{odd}\\ \text{even}\end{matrix}\right.$, and there are N + 2 thrust arcs.

II,3.3.2.3. Consumption.

a) *Development around* q = -1.

Appendix 12 shows that for $-1 \leq q \leq -\sqrt{3}/2$, we get all the optimal solutions for the angle of transfer included between 0 and 2 Arc cos (1/3) (for impulsional solutions) or even 2 Arc cos (-1/3) (for the "continuous thrust" solution). For the total of these solutions, let us posit: /105

$$q = -1 + \varepsilon \tag{43}$$

with:

$$0 \ll \varepsilon \ll 1 - \frac{\sqrt{3}}{2} = 0,134. \tag{44}$$

From (32) and (33), we derive:

$$\frac{\Delta a}{F_{max}} = A = A - B = \int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{F_{max}} \frac{-3\cos^2 L + 4\cos L(1-q) + 4q - 1}{\sqrt{3\cos^2 L + 8q\cos L + 4q^2 + 1}} dL =$$

$$\int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{F_{max}} \Big[-(1-\cos L)^{1/2}(5-3\cos L)^{-1/2} + \tag{45}$$

$$2(1+\cos L)(1-\cos L)^{-1/2}(5-3\cos L)^{-3/2}\varepsilon^2 + \text{order } \varepsilon^3\Big] dL.$$

Changing the variable (38), it becomes:

$$\frac{\Delta a}{F_{max}} = 16\left[\frac{1}{\sqrt{3}} \text{Arc tg}\left(\sqrt{3}y\right) + \frac{y}{1+3y^2}\right]_{y_1}^{y_0} + \text{order } \varepsilon^2. \tag{46}$$

Now, the equation of consumption is written:

$$\lambda_{c1} = \frac{\Delta C}{F_{max}} = \frac{W/\Delta m/}{F_{max}} = \int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{F_{max}} dL = 2(L_1 - L_0) = \Delta L + 2L_1. \tag{47}$$

The result is that:

$$\begin{cases} y_0 = \left[\dfrac{1+\cos\dfrac{\Delta L}{2}}{5-3\cos\dfrac{\Delta L}{2}}\right]^{1/2} \\ \\ y_1 = \left[\dfrac{1+\cos\left(\dfrac{\Delta L}{2}-\dfrac{W|\Delta m|}{2F_{max}}\right)}{5-3\cos\left(\dfrac{\Delta L}{2}-\dfrac{W|\Delta m|}{2F_{max}}\right)}\right]^{1/2} \end{cases} \tag{48}$$

It is important to note that in development (46) *no term in* ε *occurs*. If we make a decision to overlook terms of the order of ε^2($< (0.134)^2 = 0.018$), formula (46) reduced to its first term is an *explicit* formula permitting a calculation of the transfer Δa which can be achieved with a consumption $|\Delta m|$. /106

The function:

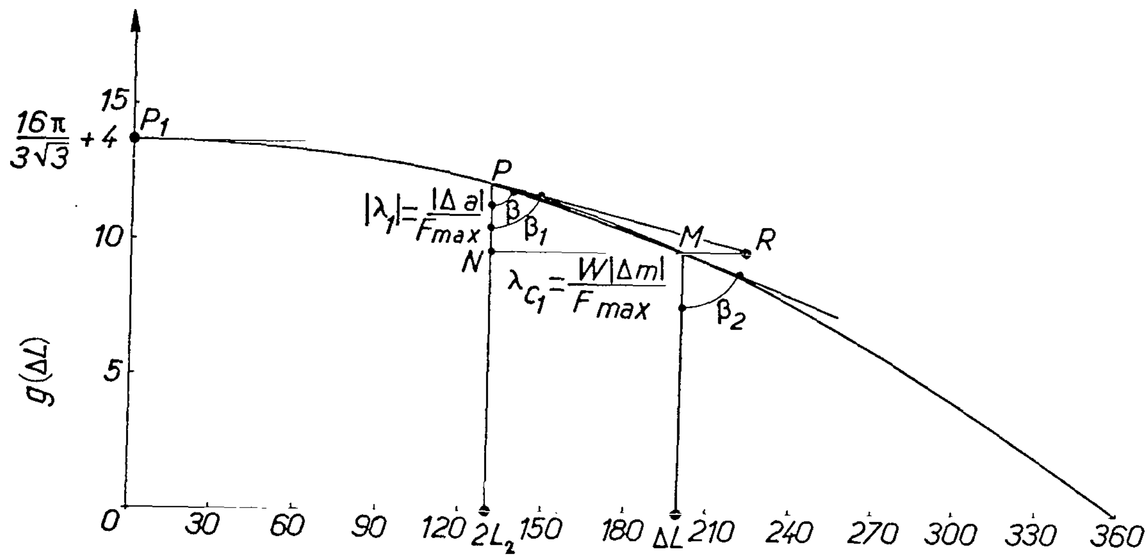
$$g(\Delta L) = 16\left[\frac{1}{\sqrt{3}}\ \mathrm{Arc\ tg}\ (\sqrt{3}y) + \frac{y}{1+3y^2}\right] \text{ with } y = \left[\frac{1+\cos\dfrac{\Delta L}{2}}{5-3\cos\dfrac{\Delta L}{2}}\right]^{1/2} \tag{49}$$

is tabulated precisely in the table below and plotted on Figure 14.

ΔL degrees	g (ΔL)	ΔL degrees	g(ΔL)
0	13,673	200	9,725
20	13,642	220	8,784
40	13,549	240	7,742
60	13,387	260	6,607
80	13,150	280	5,391
100	12,829	300	4,106
120	12,416	320	2,767
140	11,903	340	1,393
160	11,286	360	0,000
180	10,560		

Knowing the transfer angle ΔL, we get point M. Then we bring in $NM = \frac{W|\Delta m|}{F_{max}}$. Then the transfer achievable with consumption $|\Delta m|$ is given by

$$PN = \frac{\Delta a}{F_{max}} < 0.$$



Transfer angle ΔL (degrees)

Fig. 14. Curve $g(\Delta L)$.

The "specific dilatation" is given by: /107

$$|\nu| = \frac{|\lambda_1|}{\lambda_{c_1}} = \frac{|\Delta a|}{W|\Delta m|} = \frac{NP}{NM} = \mathrm{cotg}\,\beta . \tag{50}$$

It is maximal when P is in M, that is to say for bi-impulsional solutions.

Figure 15 furnishes the values of the "specific dilatation" $|\nu|$ as a function of the transfer angle for different values of the parameter

$$|\lambda_1| = \frac{|\Delta a|}{F_{max}} .$$

Bi-impulsional solutions ($\lambda_1 = 0$, Arc $\widehat{OQ}$).

In this case:

$$|\nu|_{\text{approximate}} = \mathrm{cotg}\,\beta = -\frac{dg(\Delta L)}{d(\Delta L)} = \left(1 - \cos\frac{\Delta L}{2}\right)^{1/2}\left(5 - 3\cos\frac{\Delta L}{2}\right)^{1/2} . \tag{51}$$

The curve can be graduated in values of q. As a matter of fact equation (33) is written:

$$0 = B \equiv \int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{F_{max}} \left[\begin{array}{l} (1-3\cos L)(1-\cos L)^{1/2}(5-3\cos L)^{-1/2} \\ + 4\varepsilon(1+\cos L)(5-3\cos L)^{-3/2}(1-\cos L)^{-1/2} + \text{order } \varepsilon^2 \end{array} \right] dL \qquad (52)$$

or, for bi-impulsional solutions: /108

$$0 = \frac{dB}{d(\Delta L)} = \left(1 - 3\cos\frac{\Delta L}{2}\right)\left(1-\cos\frac{\Delta L}{2}\right)^{1/2}\left(5 - 3\cos\frac{\Delta L}{2}\right)^{-1/2}$$
$$+ 4\varepsilon\left(1+\cos\frac{\Delta L}{2}\right)\left(5-3\cos\frac{\Delta L}{2}\right)^{-3/2}\left(1-\cos\frac{\Delta L}{2}\right)^{-1/2} + \text{order } \varepsilon^2.$$

An approximate value of ε is therefore:

$$\varepsilon = q + 1 \simeq -\frac{1}{4}\left(1 - 3\cos\frac{\Delta L}{2}\right)\left(1-\cos\frac{\Delta L}{2}\right)\left(5-3\cos\frac{\Delta L}{2}\right)\left(1+\cos\frac{\Delta L}{2}\right)^{-1}. \qquad (53)$$

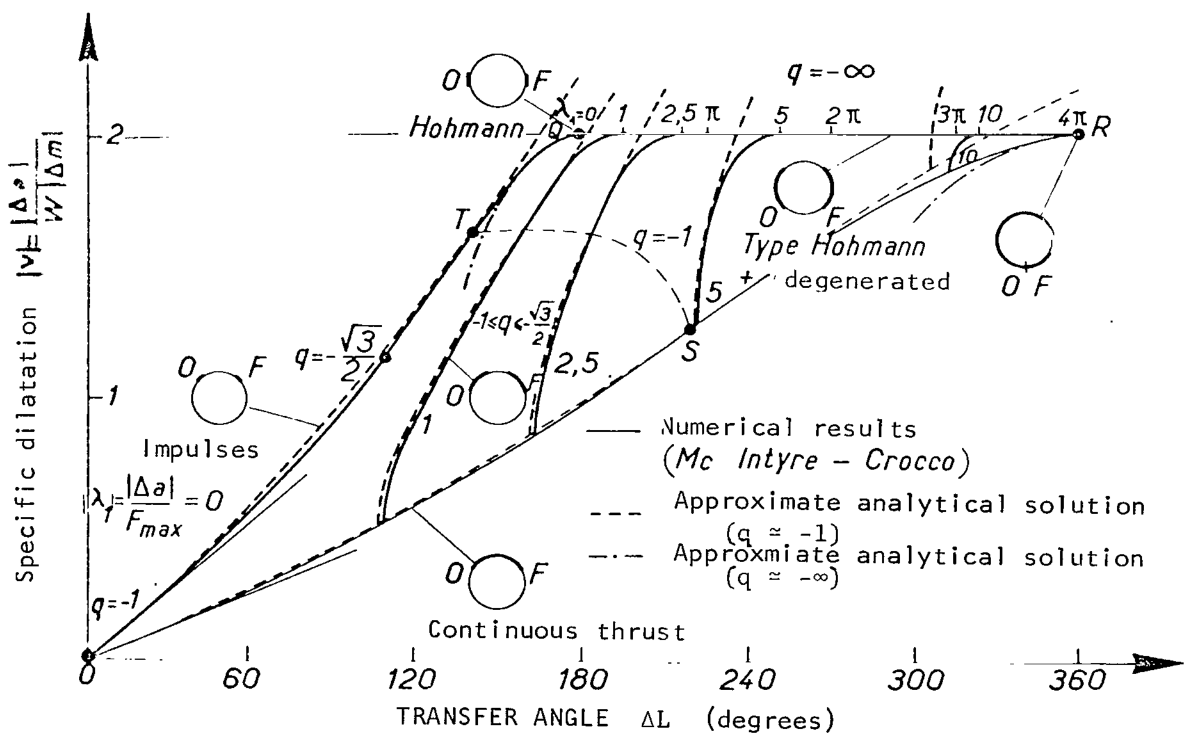


Fig. 15. Propulsion System (S_1) Performance.

The preceding results can be compared with those furnished by the following *direct calculation.*

The application of an impulse forming angle ψ with the horizontal produces:

$$\frac{\Delta a}{F_{max}\,\Delta t} = K_{ay}\cos\psi = 2\cos\psi \tag{54}$$

$$\frac{\Delta\alpha}{F_{max}\,\Delta t} = K_{\alpha x}\sin\psi + K_{\alpha y}\cos\psi = \sin L\,\sin\psi + 2\cos L\,\cos\psi\,. \tag{55}$$

Condition $\Delta\alpha = 0$ is written:

$$tg\,\psi = -\,2\,cotg\,L \quad ou \quad |\cos\psi| = \left(1 + 4\,cotg^2 L\right)^{-1/2}. \tag{56}$$

Therefore the "specific dilatation" is:

$$|\nu| \text{ calculated } = \frac{|\Delta a|}{W|\Delta m|} = 2|\cos\psi| = \left(\frac{1}{4} + cotg^2\frac{\Delta L}{2}\right)^{-1/2}. \tag{57}$$

The values of $|\nu|$ given by (51) and (57) coincide exactly for $\Delta L = 0$ ($|\nu| = 0$) and $\Delta L = 2$ Arc cos $(1/3)$ ($|\nu| = \frac{2\sqrt{2}}{3}$) because, then, $\varepsilon = 0$.

Moreover, the two curves $|\nu\,(\Delta L)|$ *are tangent* in these points, as the calculation of the derivative shows:

$$\frac{d|\nu|_{approx.}}{d(\Delta L)} = \frac{1}{2}\sin\frac{\Delta L}{2}\left(4 - 3\cos\frac{\Delta L}{2}\right)\left(1 - \cos\frac{\Delta L}{2}\right)^{-1/2}\left(5 - 3\cos\frac{\Delta L}{2}\right)^{-1/2}$$

$$\frac{d|\nu|_{calc.}}{d(\Delta L)} = 4\left(1 + 3\cos^2\frac{\Delta L}{2}\right)^{-3/2}\cos\Delta L\,.$$

The two curves are very close to each other, except obviously when the value $\Delta L = 2$ Arc cos $\frac{1}{3}$ is exceeded and when the value $\Delta L = \pi$ is approached, where q can no longer be considered as close to -1.

Let us note that:

$$\left[\frac{d|\nu|_{approx.}}{d(\Delta L)}\right]_{\Delta L=0} = \left[\frac{d|\nu|_{calc.}}{d.(\Delta L)}\right]_{\Delta L=0} = \frac{1}{2}\,. \tag{58}$$

This result can be found again very simply by a reasoning analogous to that of § II,3.3.1.: the first impulse must be made to travel the distance $|\Delta a|/2$ at a constant velocity in the period of time $\Delta L/2$ (Figure 9). The corresponding characteristic velocity is: /109

$$\Delta C_1 = \frac{|\Delta a|/2}{\Delta L/2} = \frac{|\Delta a|}{\Delta L}.$$

The total characteristic velocity is:

$$\Delta C = 2\Delta C_1 = 2\frac{|\Delta a|}{\Delta L}$$

whence

$$|\nu| = \frac{|\Delta a|}{\Delta C} = \frac{\Delta L}{2} \text{ et } \frac{|\nu|}{\Delta L} = \frac{1}{2}.$$

"Continuous thrust" solutions.

These solutions correspond to the case where point P of Figure 14 is in P_1. Therefore:

$$\lambda_1 = \frac{\Delta a}{F_{max}} = 16\left[\frac{1}{\sqrt{3}} \operatorname{Arc\,tg}(\sqrt{3}\,y_0) - \frac{\pi}{3\sqrt{3}} + \frac{y_0}{1+3y_0^2} - \frac{1}{4}\right] \tag{59}$$

with

$$y_0 = \left[\frac{1+\cos\dfrac{\Delta L}{2}}{5-3\cos\dfrac{\Delta L}{2}}\right]^{1/2}$$

Agreement with the numerical results of McIntyre and Crocco is satisfactory for $q > -1$. The approximate curve (Figure 15) and the numerically calculated curve are tangent at $O(\Delta L = 0)$ and at point $\Delta L = 2 \operatorname{Arc\,cos}(-1/3)$ corresponding to:

$$|\lambda_1| = \frac{|\Delta a|}{F_{max}} = \frac{8\pi}{3\sqrt{3}} \simeq 4,84\ ,\ |\nu| = 4\pi/3\sqrt{3}\operatorname{Arc\,cos}\left(-\frac{1}{3}\right) \doteq 1.265$$

for in the two cases, $q = -1(\varepsilon = 0)$.

Let us note that for $\Delta L \simeq 0$, formula (59) gives:

$$\frac{|\nu|}{\Delta L} \sim \frac{1}{4}. \tag{60}$$

This result can also be found very simply by reasoning analogous to that of § II,3.3.1: thrust acceleration F_{max} applied in a continuous fashion should cover the distance $|\Delta a|/2$ in the time $\Delta L/2$. Whence:

$$\frac{1}{2} F_{max} \left(\frac{\Delta L}{2}\right)^2 = \frac{|\Delta a|}{2}.$$

The corresponding characteristic velocity is:

$$\Delta C_1 = F_{max} \frac{\Delta L}{2} = \frac{2|\Delta a|}{\Delta L}.$$

The total characteristic velocity is:

$$\Delta C = 2 \Delta C_1 = \frac{4|\Delta a|}{\Delta L} \text{ with } |\nu| = \frac{|\Delta a|}{\Delta C} = \frac{\Delta L}{4}.$$

Solutions corresponding to fixed values of $|\lambda_1| = \frac{|\Delta a|}{F_{max}}$.

In Figure 15 have been plotted the lines corresponding to the values $|\lambda_1| = 0, 1, 2.5, 5, 10$.

Agreement with the numerical results of McIntyre and Crocco remains satisfactory before reaching the line $\widehat{TS}$ corresponding to $q = -1$. This line can be plotted approximately. /110

CONCLUSION: in the entire OTS domain and a little beyond the $\widehat{TS}$ frontier, formula (46) reduced to its first term can be used if an error of the order of 2% is admitted.

On the other hand, in the area of singular solutions, this approximate solution is far from the numerical solution and it is necessary to have recourse to another development.

b) *Development around* $q = -\infty$.

Let us posit:

$$q = -\frac{1}{\varepsilon_1} \quad (0 < \varepsilon_1 \ll 1). \tag{61}$$

Carried into (32) and (33) it becomes:

$$\lambda_1 = \frac{\Delta a}{F_{max}} = A = \int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{F_{max}} \left[-2 + \frac{\varepsilon_1^2}{4} \sin^2 L + \text{order } \varepsilon_1^3 \right] dL \tag{62}$$

$$- 4 \delta L + \frac{\varepsilon_1^2}{2} \delta \left(\frac{L}{2} - \frac{\sin 2L}{4} \right) + \text{order } \varepsilon_1^3$$

$$B = 0 = \int_{-\frac{\Delta L}{2}}^{+\frac{\Delta L}{2}} \frac{F}{F_{max}} \left(- 2 \cos L + \frac{\varepsilon_1}{2} \sin^2 L + \text{order } \varepsilon_1^3 \right) dL = \tag{63}$$

$$- 2 \delta \sin L + \frac{\varepsilon_1}{2} \delta \left(\frac{L}{2} - \frac{\sin 2L}{4} \right) + \text{order } \varepsilon_1^3 .$$

On the other hand:

$$\dot{\lambda}_{c_1} = \frac{W/\Delta m/}{F_{max}} = 2 \delta L \tag{64}$$

where the sign δ signifies that the difference of the quantities envisaged for the values L_f and L_2 of Figure 12 must be taken.

Deriving ε_1 from (63):

$$\varepsilon_1 = \frac{8 \dfrac{\delta \sin L}{\delta L}}{1 - \dfrac{\delta \sin 2L}{\delta (2L)}} + \text{order } \varepsilon_1^2 \tag{65}$$

and introducing it into (62) it becomes:

$$\lambda_1 = \frac{\Delta a}{F_{max}} = - 4 \delta L \left[1 - 4 \frac{\left(\dfrac{\delta \sin L}{\delta L} \right)^2}{1 - \dfrac{\delta \sin 2L}{\delta (2L)}} \right] + \text{order } \varepsilon_1^3 . \tag{66}$$

If ε_1 is sufficiently small, that is to say, according to (65), if $\sin L_2 \approx \sin L_f$ (area of the solutions of the Hohmann type), it is possible to use the following *explicit* formula while ignoring the terms of the order of ε_1^3:

$$\lambda_1 = \frac{\Delta a}{F_{max}} = -4\left[\frac{\Delta L}{2} - L_2\right]\left[1 - 4\frac{\left(\dfrac{\sin\frac{\Delta L}{2} - \sin L_2}{\frac{\Delta L}{2} - L_2}\right)^2}{1 - \dfrac{\sin \Delta L - \sin 2L_2}{\Delta L - 2L_2}}\right] \quad (67)$$ /111

with

$$L_2 = \frac{\Delta L}{2} - \frac{W|\Delta m|}{2F_{max}},$$

in order to calculate the transfer corresponding to a given consumption.

When $q \to -\infty$, $\sin L_2 = \sin L_f$ and then

$$\lambda_2 = \frac{\Delta a}{F_{max}} = -4(L_f - L_2) = -4(\Delta L - \pi).$$

The "specific dilatation" is *maximal* (Hohmann type solutions).

$$|\nu| = \frac{|\Delta a|}{W|\Delta m|} = 2. \quad (68)$$

BI-IMPULSIONAL SOLUTIONS.

These correspond to $L_2 = L_f$. Then:

$$|\nu|_{approx.} = \frac{|\Delta a|}{W|\Delta m|} = \frac{|\lambda_1|}{\lambda_{c1}} = 2\left[1 - 4\frac{\left(\dfrac{\delta \sin L}{\delta L}\right)^2}{1 - \dfrac{\delta \sin 2L}{\delta(2L)}}\right] = \quad (69)$$

$$2\left[1 - \frac{4\cos^2\frac{\Delta L}{2}}{1 - \cos\Delta L}\right] = 2\left(1 - 2\,\mathrm{cotg}^2\frac{\Delta L}{2}\right)$$

a formula to be compared with that furnished by the exact calculation taking part in (57):

$$|\nu|_{calc.} = \left(\frac{1}{4} + \mathrm{cotg}^2\frac{\Delta L}{2}\right)^{-1/2} \underset{\Delta L \to \pi}{\sim} 2\left(1 - 2\,\mathrm{cotg}^2\frac{\Delta L}{2}\right).$$

The curve can be grade' in values of ε_1:

$$\varepsilon_1 \simeq \frac{8 \cos \frac{\Delta L}{2}}{1 - \cos \Delta L} = 4 \frac{\cos \frac{\Delta L}{2}}{\sin^2 \frac{\Delta L}{2}} \tag{70}$$

or:

$$\varepsilon_1 \underset{\Delta L \to \pi}{\sim} 4 \cos \frac{\Delta L}{2}.$$

The agreement between $|\nu|_{\text{approximate}}$ and $|\nu|_{\text{calculated}}$ is good in the /112
vicinity of point Q. Then the two solutions diverge rapidly.

"CONTINUOUS THRUST" SOLUTIONS.

These solutions correspond to $L_2 = 0$. Whence:

$$|\nu|_{approx.} = \frac{|\Delta a|}{W/|\Delta m|} = 2 \left[1 - 4 \frac{\left(- \frac{\sin \frac{\Delta L}{2}}{\frac{\Delta L}{2}} \right)^2}{1 - \frac{\sin \Delta L}{\Delta L}} \right] \tag{71}$$

The corresponding values of ε_1 are:

$$\varepsilon_1 \simeq \frac{8 \frac{\sin \frac{\Delta L}{2}}{\Delta L / 2}}{1 - \frac{\sin \Delta L}{\Delta L}} \tag{72}$$

which permits the curve to be graduated in values of ε_1.

. Agreement with the numerical results of McIntyre and Crocco is good in the vicinity of point R, after which the two solutions diverge rapidly.

CONCLUSION: although the solutions corresponding to fixed values of $|\lambda_1| = \frac{|\Delta a|}{F_{max}}$ have not been tabulated except for zero ($|\lambda_1|$ = 1, 2.5, 5, 10 for example), it may reasonably be thought that in a *narrow* band in the vicinity of the singular solutions QR, formula (67) can be used as a first approximation.

Case where $\Delta L \gg 2\pi$.

In this case we have seen that $|q| \gg 1$ with the result that it is possible to use a development in the area of $|q| = +\infty$. Formula (66) can be used with a limitation to the first term and with suitable study of the significance of δ.

II,3.3.2.4. Comparison with the propulsion system (S_2): W variable, $P - P_{max}$

a) *System* (S_{1c}) *with continuous thrust.*

In Figure 8 have been traced the variations of the reduced dilatation

$$|\zeta| = \frac{|\Delta a|}{\sqrt{2 P_{max}}\, |\Delta m| \Delta L}$$

as a function of the transfer angle ΔL, *for "continuous thrust" solutions* (for these solutions $|\zeta| = |\nu|$). Thus it is possible to appreciate the penalty due to not modulating the ejection velocity W. However, contrary to the result obtained in the case of infinitesimal rotation, the penalty is 0 for $\Delta L = 2N\pi$ (whole number of revolutions), because the optimal solution obtained for the propulsion system (S_2): W variable, $P = P_{max}$, which consists of applying a constant thrust tangentially, coincides with the solution obtained by the propulsion system (S_1): $W = C^{te}$, $F \leq F_{max}$.

The slopes at the origin of the curves are respectively: /113

systèm (S_2): $\frac{|\zeta|}{\Delta L} = 1/2\sqrt{3}$ [équation (14)]

systèm (S_{1C}): $\frac{|\zeta|}{\Delta L} = 1/4$.

For $\Delta L \simeq 0$, the penalty is:

$$\frac{(1/2\sqrt{3}) - (1/4)}{1/2\sqrt{3}} = 1 - (\sqrt{3}/2) = 0{,}134$$

$\simeq$ 13% of $|\zeta|$, therefore 26% of $|\Delta m|$.

b) *System* (S_1^*) *with optimal (constant) ejection velocity* W^*.

The transfer being fixed (and in particular the transfer angle ΔL), L_2 fixes the length of the thrust arcs. Into the hypothesis we shall place $|q| \simeq 1$ (OTS region of Figure 15). Optimizing the ejection velocity W with constant P_{max} power is the same as optimizing L_2, i.e. seeking the maximum of:

$$|\zeta(L_2)| = \sqrt{\lambda \nu} = \frac{|\lambda|}{\sqrt{\lambda_c}} \qquad \left(\lambda = \frac{\Delta a}{F_{max}\, \Delta L} = \frac{\lambda_1}{\Delta L}\right)$$

where $|\lambda_1|$ is given by (45) and where $\lambda_c = 1 - \frac{2L_2}{\Delta L}$.

Now, $\frac{d|\zeta|}{dL_2}$ is zero for:

$$2\lambda_c \frac{d|\lambda|}{dL_2} = |\lambda| \frac{d\lambda_c}{dL_2} = -\frac{2|\lambda|}{\Delta L}.$$

Thus:

$$|\nu| = \frac{|\lambda|}{\lambda_c} = -\Delta L \frac{d|\lambda|}{dL_2} = -\frac{d|\lambda_1|}{dL_2} \simeq 2(1-\cos L_2)^{1/2}(5-3\cos L_2)^{-1/2}.$$

In Figure 14 this condition is expressed by the very simple condition:

$$\cot g\, \beta^* = 2 \cot g\, \beta_1^* \quad \text{or} \quad \boxed{NM^* = MR^*} \tag{73}$$

which easily furnishes the optimal length of the thrust arcs.

When the transfer angle is weak ($\Delta L \sim 0$), L_2 is also weak and

$$\cot g\, \beta \simeq \frac{\cot g\, \beta_1 + \cot g\, \beta_2}{2}$$

where $\cot g\, \beta_2$ is the value of $|\nu|$ relative to the bi-impulsional solution of transfer angle ΔL.

From this is deduced:

$$\cot g\, \beta = \frac{2}{3} \cot g\, \beta_2 \left(= 2 \cot g\, \beta_1\right). \tag{74}$$

In Figure 15 the slope at the origin of the curve corresponding to the system (S_1^*) is therefore:

$$\cot g\, \beta^* = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} \tag{75}$$

On the other hand, /114

$$\cot g\, \beta_1 = \frac{(\Delta L - 2L_2)\cot g\, \beta_0 + 2L_2 \cot g\, \beta_2}{\Delta L} = 2\frac{L_2}{\Delta L} \cot g\, \beta_2.$$

Thus, combined with (74),

$$L_2^* = \frac{\Delta L}{6}. \tag{76}$$

The result is that:

$$|\zeta^*| = \sqrt{\lambda \nu} = |\nu| \sqrt{\lambda_c} = \frac{\Delta L}{3}\sqrt{1 - \frac{2L_2^*}{\Delta L}} = \frac{\Delta L \sqrt{2}}{3\sqrt{3}} \tag{77}$$

which, in Figure 8, furnishes the slope at the origin of the curve relative to the system (S_1^*).

The preceding results can be found again rapidly by a direct reasoning based on Figure 9.

Condition (I,3 - 80) is expressed by the equality of the areas (1) and (2) (Figure 16). Therefore $2L_2^* = \frac{1}{3}\frac{\Delta L}{2}$

The penalty due to not modulating the ejection velocity in this case is no more than:

$$\frac{\Delta|\zeta|}{|\zeta|} = \frac{\frac{1}{2\sqrt{3}} - \frac{1}{3}\sqrt{\frac{2}{3}}}{\frac{1}{2\sqrt{3}}} = 0{,}059$$

or about 12% of mass consumption instead of 26%.

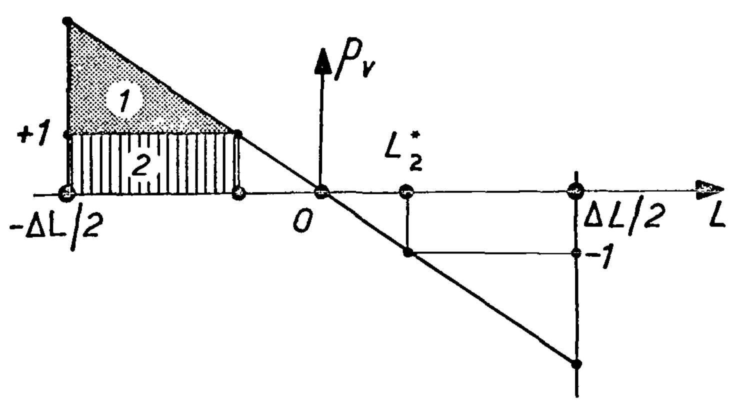


Fig. 16. Optimal Length of Maximal Thrust Arcs. (Very Small Transfer Angle) (System S_1^*).

II,3.3.2.5. Singular solutions of the linearized problem. Degeneracy.

We have seen that the solutions corresponding to $p_a = -\frac{1}{2}$, $p_e = 0^+$, therefore $q = -\infty$, are singular (type I bis).

As a matter of fact, since the commutation function $\Theta = |\vec{p}_V| - 1$ is identically zero, it is impossible to determine the thrust modulus, which nevertheless remains tangential, with the sole application of the Maximum Principle.

However (27) and (30) are always valid:

$$|\lambda_1| = \frac{|\Delta a|}{F_{max}} = |A| = 2\int_{-\frac{\Delta L}{2} - L_e}^{+\frac{\Delta L}{2} - L_e} \frac{F}{F_{max}}\, dL \tag{78}$$

$$\int_{-\frac{\Delta L}{2} - L_e}^{+\frac{\Delta L}{2} - L_e} \frac{F}{F_{max}} \cos L\, dL = 0 \tag{79}$$ /115

$$\int_{-\frac{\Delta L}{2} - L_e}^{+\frac{\Delta L}{2} - L_e} \frac{F}{F_{max}} \sin L\, dL = 0. \tag{80}$$

These equations receive an interpretation analogous to that already met in § II,1.3.2.2.:

The singular solutions corresponding to a given value of $\lambda_1 = \Delta a/F_{max}$ are such that the thrust is applied tangentially ($|q| = \infty$) and modulated ($F = F(L)$) while respecting only the following rule:

The "mass" $|\Delta a|/2$ is distributed on the transfer arc $\widehat{M_0 M_f}$ with the "linear density" $F = F(L) \leq F_{max}$ *in such a way that the center of the mass is at* 0.

There does not exist any singular solution corresponding to the value $|\lambda_1|$ given for a transfer angle ΔL less than a limit value:

$$\Delta L(\lambda_1) = \pi + \frac{|\lambda_1|}{4} > \pi$$

for which the only solution corresponding to a tangential thrust is the solution of the Hohmann type.

For $\Delta L(\lambda_1) < \Delta L < 2\pi$, and among the singular solutions, there can be envisaged (Figure 17): solutions of the Hohmann type, inclined solutions of the Hohmann type, solutions which are deduced from them by sectioning, and finally solutions where the thrust is modulated on certain arcs.

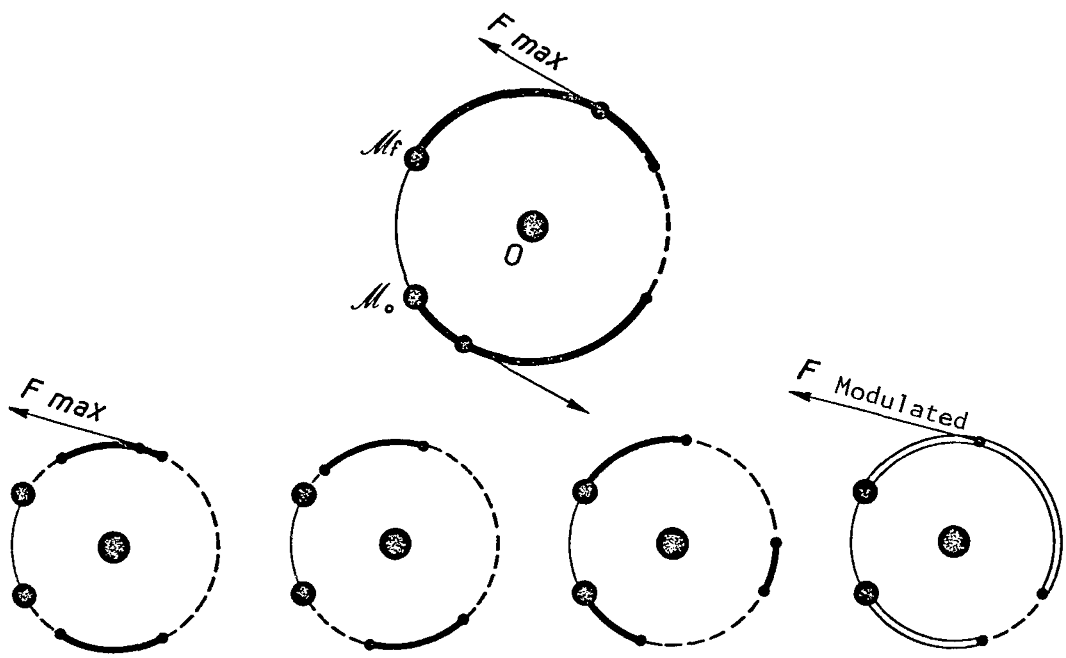


Fig. 17. Hohmann Type Solution and Singular Solutions.

The preceding results could be extended to the case $\Delta L > 2\pi$. /116

In order to remove the degeneracy, recourse must be had to a higher order calculation which will be explained in Chapter II,4.

II,3.4. Conclusion.

The analytical study of the transfer problem between infinitely close, co-planar circular orbits, with the transfer angle being fixed at any value, proves to be, as expected, much more delicate for the propulsion system (S_1) with constant ejection velocity and limited thrust than for the propulsion system (S_2) with variable ejection velocity and limited jet power.

However, the adoption of the orbital elements a, α, β as state coordinates greatly simplifies the calculations. In particular it is possible to construct geometrically the optimal thrust at any point of the optimal trajectory and to calculate consumption in a rigorous manner for the system (S_2) and in an approximate manner for the system (S_1).

*

In the case of the propulsion system (S_2) (variable ejection velocity, limited jet power), the thrust is modulated in a continuous fashion along the transfer arcs. It is antisymmetrical in relation to the axis of symmetry of the transfer arc.

The law of thrust orientation depends only on the transfer angle. For weak transfer angles (up to about 187°), the tangential component of the thrust changes direction twice. In the case of a whole number of revolutions (or of a great number of revolutions) the optimal thrust is tangential and constant.

The reduced dilatation starts from the zero value for the zero transfer angle and after a few oscillations stabilizes at a maximal value, a value which furthermore is attained without being exceeded in the case of a whole number of revolutions. This value corresponds to the reduced dilatation which would be obtained for any transfer angle if the final orbit was not obliged to be circular.

*

In the case of the propulsion system (S_1) (constant ejection velocity, limited thrust), a distinction must be made between the regular solutions and the singular solutions which can only appear for a transfer angle greater than 180°.

The regular solutions entail an alternation of maximal thrust arcs and of ballistic arcs, the first arc and the last arc always being propulsed arcs. The optimal thrust is antisymmetrical in relation to the axis of symmetry of the transfer arc. The law of thrust orientation depends not only on the transfer angle but also on the relationships between the relative dilatation of the radius of the orbit and the maximal thrust available related to Newtonian attraction.

Since the angle of transfer and the dilatation to be achieved are fixed, there exists a minimal value of maximal thrust available below which transfer is impossible.

This value corresponds to the transfer where the thrust is applied continuously (no ballistic arc).

When the maximal thrust available increases beyond this value, one or several ballistic arcs appear and gradually expand, which permits the thrust to be localized in zones where efficiency is better. Then specific dilatation increases up to a maximal value.

If the transfer angle is less than 180°, this maximal value is lower than 2 and corresponds to a bi-impulsional solution (the propulsed arcs are reduced to two points). The maximal thrust related to local attraction is then very large in regard to the relative dilatation to be achieved.

If the transfer angle is greater than 180°, the maximal value is equal to 2 and corresponds to Hohmann type solutions (two maximal thrust arcs symmetrical in relation to origin) for a transfer angle lower than 360°, or of the generalized Hohmann type for a transfer angle above 360°. Thrust is then tangential. /117

If the maximal thrust available still increases beyond the value corresponding to this type of solution (which therefore implies that the transfer angle is greater than 180°), the solution of the first order is no longer unique and degenerates into an infinity of singular solutions.

The singular solutions correspond to a tangential application of thrust, whether modulated or not. It is enough to distribute the thrust along the transfer arc in such a way that the desired dilatation is obtained and that the center of mass of the distribution is at the center of the orbit. These singular solutions are particularly important because, both for them, for the Hohmann solution and for solutions of the Hohmann and Hohmann generalized types specific dilatation is maximal (at least in the linearized study).

Between two propulsion systems of the same power functioning in a continuous fashion, consumption is obviously less for the one with an ejection velocity which can be modulated.

The penalty in consumption due to not modulating the ejection velocity is of the order of 26% for a continuous thrust system or of 12% for an optimal (constant) ejection velocity system in the case of a weak transfer angle.

II,4. PLANE OPTIMAL TRANSFERS OF THE HOHMANN TYPE BETWEEN NON-INTERSECTING DIRECT, COAXIAL, CLOSE NEAR-CIRCULAR ORBITS.

(Propulsion system (S_1) with constant ejection velocity and limited thrust)

II,4.1. INTRODUCTION.

In § II,3.3.2.5. on the linearized study of optimal transfers between close, coplanar, circular orbits, when the transfer angle is fixed and for propulsion systems (S_1), we have shown the existence of solutions which lead to a propulsive consumption equal (on the order under consideration) to that of Hohmann's bi-impulsional solution, although they entail continuous thrust arcs.

The higher order study of these solutions has been made by McIntyre and Crocco [42, 44] by applying Pontryagin's Maximum Principle, which leads to rather complex calculations. Here we shall use a direct analytical method and the results obtained will not only concern the cases of plane transfers between close, coplanar, circular orbits, but also generally that of plane transfers between non-intersecting, direct, coaxial, close, near-circular orbits.

II,4.2. REVIEW OF THE RESULTS OF THE LINEARIZED STUDY.

The osculating orbit $(\mathcal{O})$ is defined by its "perigee vector" $\vec{e}$, directed toward the perigee P and of length e = order ε, with the components:

$$\vec{e} \begin{cases} \alpha = e \cos \varpi \\ \beta = e \sin \varpi \end{cases} \qquad (1)$$

on fixed axes $\vec{Ox}$ and $\vec{Oy}$ (Figure 1) and by the quantity: /118

$$B = \sqrt{\frac{\mu}{b}} \qquad (b = \text{semi-minor axis} = a\sqrt{1 - e^2}). \qquad (2)$$

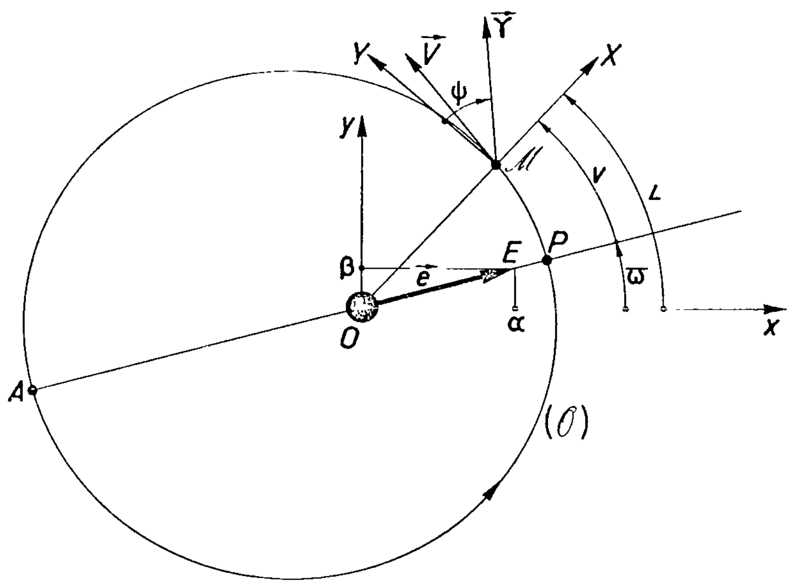


Fig. 1. Perigee Vector.

The utilization of B, instead of semi-major axis a, in this study is justified by the fact that the variation dB of B requires a characteristic velocity dC equal to -dB with a relative error of the *second order* only in relationship to:

$$M = max(\varepsilon, \psi) = \text{ infinitely small principal.} \qquad (3)$$

The application of the thrust acceleration $\vec{\gamma}$ in the direction forming the angle $\psi \simeq 0$ with the local horizontal produces these variations:

$$\frac{d\alpha}{dC} = 2\cos L + \text{order } M \tag{4}$$

$$\frac{d\beta}{dC} = 2\sin L + \text{order } M \tag{5}$$

$$\frac{dB}{dC} = -1 + \frac{1}{2}\left(\varphi - \frac{e}{2}\sin v\right)^2 - \frac{3e^2}{8}\left(1+\cos^2 v\right) + \text{order } M^3 \tag{6}$$

which are easily deduced from the traditional formulae of perturbations and from the definition of the characteristic velocity C:

$$C = \int_{t_0}^{t} \gamma\, dt \qquad (\gamma = |\vec{\gamma}|) \tag{7}$$

which takes the place of "consumption" for propulsion systems with constant ejection velocity W or a known function of the mass m of the mobile.

v is the true anomaly, $L = \bar{\omega} + v$ the straight ascent.

The present study applies to propulsion systems with limited acceleration ($0 \leq \gamma \leq \gamma_{max}$ fixed). It also extends without modification (except where otherwise indicated) to propulsion systems with limited thrust ($F \leq F_{max}$ fixed) if the ejection velocity W is of the order of the nominal velocity in orbit V, /119 which assures that the relative mass variation is of the order ε. Then it is enough, supposing that the unity of mass is equal to the initial mass m_0 of the mobile, to replace γ_{max} by F_{max} in the calculations.

Taking B instead of C as an independent variable in equations (4), (5) and (6), we get:

$$\frac{d\alpha}{-dB} = 2\cos L + \text{order } M \tag{8}$$

$$\frac{d\beta}{-dB} = 2\sin L + \text{order } M \tag{9}$$

$$-\frac{dC}{dB} = 1 + \frac{1}{2}\left(\varphi - \frac{e}{2}\sin v\right)^2 - \frac{3e^2}{8}\left(1+\cos^2 v\right) + \text{order} M^3 \tag{10}$$

and by integration:

$$\Delta\alpha = \int_{B_0}^{B_f} 2\cos L\,(-dB) + \text{order } M^2 \tag{11}$$

$$\Delta\beta = \int_{B_0}^{B_f} 2\sin L\,(-dB) + \text{order } M^2 \tag{12}$$

$$\Delta C = -\Delta B + \text{order} M^3. \tag{13}$$

Let us then define the law of variation in optimal thrust magnitude in the following manner:

1. we distribute on arc $\Delta L = \Delta t\ (1 + \text{order}\ \varepsilon)$ (Δt fixed = duration of transfer) of the circle (O) of a unit radius, the fictitious, total, fixed "mass" $-\Delta B > 0$, with the distribution function $-[B(L) - B_0]$ so that the center of gravity of this distribution is situated at the fixed point G ,

$$G \begin{cases} x_G = \dfrac{\Delta\alpha}{-2\Delta B} \\ y_G = \dfrac{\Delta\beta}{-2\Delta B} \end{cases} \tag{14}$$

and that the linear density $\frac{-dB}{dL} \simeq \gamma$ is $\leq \gamma_{max}$.

G is necessarily inside the circle (O), and therefore $|\vec{\Delta e}| \leq 2|\Delta B|$. Therefore the initial orbit and the final orbit are not intersecting.

2. the law of variation of thrust magnitude is chosen in such a way that the characteristic velocity of consumption from the departure to position L is exactly equal to the distribution function defined above (at approximately the order M^3), that is to say:

$$C(L) = -[B(L) - B_0] + \text{order} M^3.$$

This thrust law verifies equations (11), (12) and (13), that is to say it achieves transfer at approximately the order M^2 and it is possible to adjust the transfer angle ΔL so that the duration is equal to Δt.

Consumption relative to any two of these solutions obtained by choosing distributions of different masses,but achieving the same transfer at approximately the order M^2,differs only by the order of M^3 as equation (13) shows. If only the terms of the first order in M are retained, the solutions are equivalent. The great freedom which then exists in general in the choice of the distribution B(L) is expressed by a *degeneracy* of the solution of the order M. /120

If we insert into the terms of order M^2 of equation (10) the values of e and v relating to the solution of order M, we commit a relative error of the order of M, therefore an absolute error of order M^3, which is negligible.

Now, the values of e and v of the solution of order M do not depend on the angle ψ. Therefore,the term in $e^2(1 + \cos^2 v)$ of equation (10) is independent of ψ and we see that the value of ψ, which minimizes the consumption ΔC for a fixed variation ΔB, is:

$$\psi = \frac{e}{2}\sin v + \text{order } \varepsilon^{3/2}. \tag{15}$$

A primary consequence is that ψ = order ε, therefore $M = \varepsilon$ and the preceding study of the order M coincides with the linearized study of the problem. (§ II,3.3.2.5.).

We now propose here, through a study of orders higher than the first, to remove the indetermination by selecting the optimal solutions from among the degenerated solutions of the linearized problem.

II,4.3. HIGHER ORDER STUDY.

Equation (15) shows that the optimal thrust must be applied forward, practically following the interior bisectrix of the angle $\widehat{YMV}$ between the local horizontal $\overrightarrow{MY}$ and the tangent to the trajectory $\overrightarrow{MV}$ (Figure 1).

Having made this choice, integration of (10) furnishes:

$$\Delta C = -\Delta B - \frac{3}{8}\int_{B_0}^{B_f} e^2(1+\cos^2 v)(-dB) + \text{order } \varepsilon^4. \tag{16}$$

The integral occurring in the second member is written:

$$\dot{I} = \int_{B_0}^{B_f} e^2(1+\cos^2 v)(-dB) = \frac{1}{2}\int_{(\mathcal{L})}\left[\vec{e}^{\,2} + (\vec{e}\cdot\vec{t})^2\right] ds \tag{17}$$

where $\vec{t}$ is the unitary vector of the tangent to the locus (L) of the extremity E of the perigee vector $\vec{e}$ and s is the curvilinear abscissa on this locus (Figure 2). The straight ascent L is nothing but the angle $L = (\overrightarrow{Ox}, \overrightarrow{Et})$, and the true anomaly the angle $v = (\overrightarrow{OE}, \overrightarrow{Et})$. The transfer angle ΔL is the angle swept by the tangent $\overrightarrow{Et}$. Let us note that it is enough for each of these elements to be defined with a relative precision of the order ε. /121

The problem comes down to determining the curve (L) of fixed length:

$$S = \int_{(\mathcal{L})} ds = -2\Delta B \geqslant |\overrightarrow{\Delta e}| = E_0 E_f \geqslant 0 \tag{18}$$

which renders the integral $\dot{I}$ maximal.

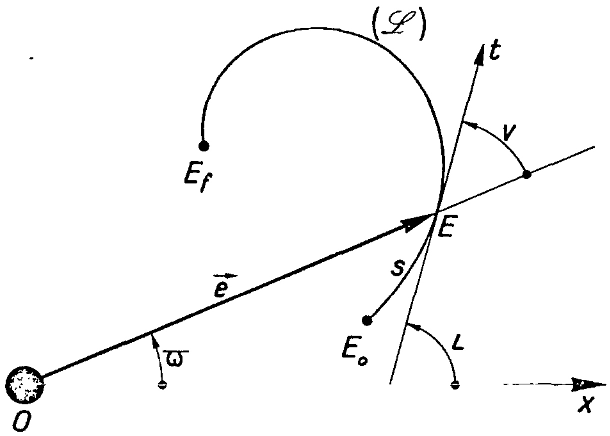


Fig. 2. Locus of the Extremity of the Perigee Vector.

Without restricting the generality of the study, we can suppose that the acceleration is equal to the maximal acceleration γ_{max} or is zero, any possible intermediate acceleration phase ($0 < \gamma < \gamma_{max}$) being theoretically able to be considered as a rapid succession of maximal thrust arcs and ballistic arcs.

The integration of (3) and (4) on a maximal acceleration arc furnishes:

$$E \begin{cases} \alpha = \alpha_i + 2\,\gamma_{max}\,(\sin L - \sin L_i) + \text{order } \varepsilon^2 \\ \beta = \beta_i - 2\,\gamma_{max}\,(\cos L - \cos L_i) + \text{order } \varepsilon^2 \end{cases} \tag{19}$$

Therefore the point E describes, at approximately order ε^2, an arc of circle (s_k) with a radius $\rho = 2\gamma_{max}$ (Figure 3). Therefore the problem consists in adjusting these "festoons" so that their total length $\sum_{k=1}^{n} = S$ is fixed and the integral I is minimal.

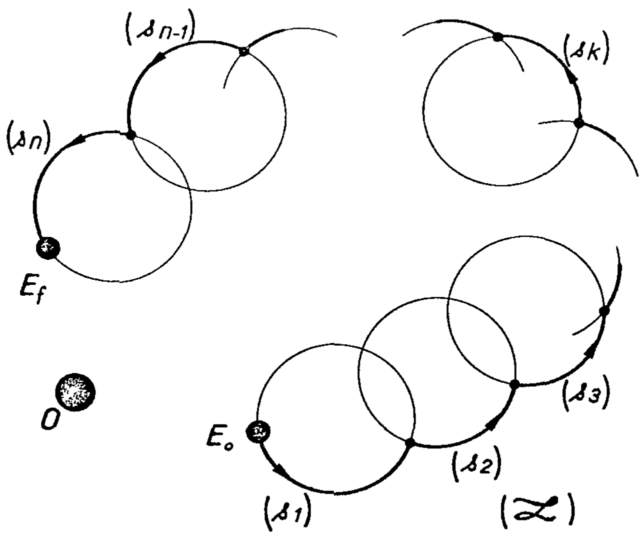


Fig. 3. Succession of Maximal Acceleration Arcs and Ballistic Arcs.

II,4.3.1. Impulsional Solutions.

If the maximal acceleration γ_{max} available is infinite, the festoons are reduced to segments (radius $\rho = \infty$). The straight ascent L is constant when the thrust is applied. Therefore there is a succession of impulses (Figure 4). The general impulsional case (any points E_0 and E_f.) has been studied by Marchal [15]. Here we shall limit ourselves to the case of transfers between direct coaxial orbits. The points 0, E_0 and E_f are then aligned and 0 is outside the segment E_0E_f(Figure 5).

We still have: /122

$$\dot{I} = \frac{1}{2}\int_{(\mathscr{L})} e^2(1+\cos^2 v)\,ds \leqslant \int_{(\mathscr{L})} e^2\,ds = \dot{I}'. \quad (20)$$

Let us suppose that we have found the route (L_H) of length S which maximizes $\dot{I}'$ and that under these conditions $\cos v \equiv 1$. (L_H) then maximizes $\dot{I}$. Now:

$$e \leqslant e_0 + s \quad (21)$$

$$e \leqslant e_f + S - s \quad (22)$$

(Figure 6)

$\dot{I}'$ is therefore maximal when the function e(s) is represented by the line ABC (Figure 7) i.e. when (L_H) is formed of two segments E_0M and ME_f borne by the straight line OE_0E_f, with M being the point of (L_H) furtherest removed from the origin O (Figure 8).

Thus we get $\cos v \equiv 1$, therefore, $\dot{I} = \dot{I}'$ is maximal.

The optimal solution consists of applying an impulse to the perigee ($v = 0$) of the initial orbit and an impulse to the apogee ($v = \pi$) of the final orbit. /123

(Hohmann transfer) (Figure 9).

If there are several revolutions available, it is possible to break the segments E_0M and ME_f into sub-segments:

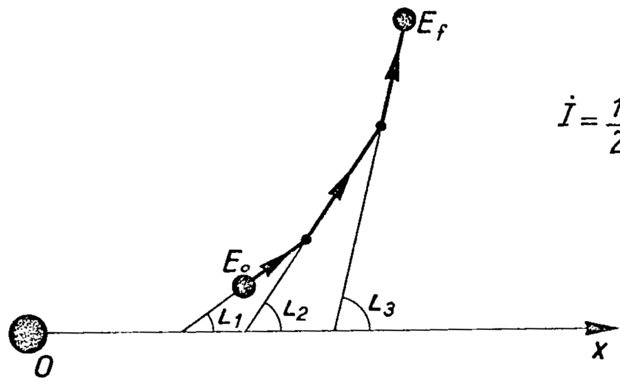


Fig. 4. Impulsional Case.

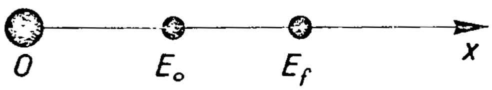


Fig. 5. Direct Coaxial Orbits.

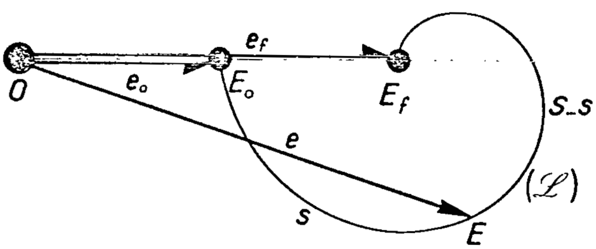


Fig.6.Non-intersecting,Direct Coaxial Orbits.

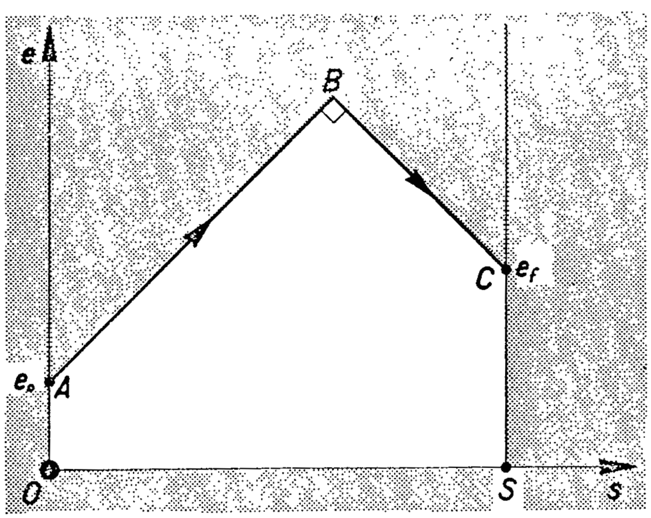


Fig. 7.

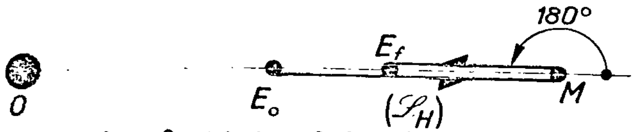


Fig. 8. Bi-impulsional Transfer.

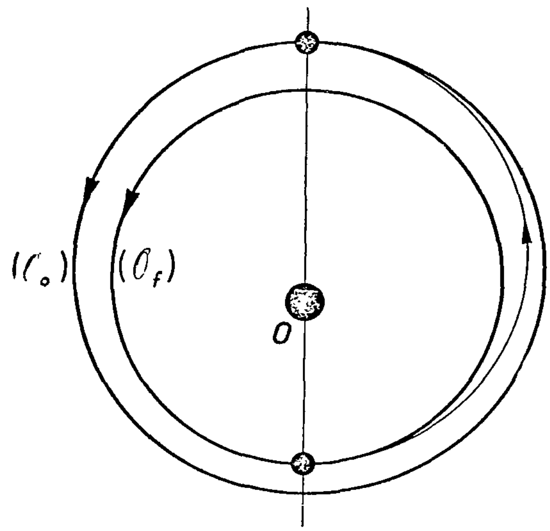


Fig. 9. Bi-impulsional Transfer.

E_0E_1, E_1E_2, E_2M, ME_3, E_3E_f, for example (Figure 10), i.e. to fraction the preceding impulses (Figure 11); then we can describe a whole number n_k of times each intermediate orbit is represented by the point E_k. [The tangent to (L_H) and E_k sweeps an angle of n_k 360°].

In any case the impulses are first applied to the perigee, then to the apogee. There is no alternation [(L_H) has only one point of retrogression: in M]. These traditional results particularly take part in [11].

II,4.3.2. Limited Acceleration Solutions. /124

Since the maximal acceleration γ_{max} is no longer infinite, it is impossible to have cos $v \equiv 1$ along the route (L) composed of festoons. Equation (16) and the inequality (20) then shows that for the same transfer consumption ΔC, relative to a limited acceleration solution, even optimal, is greater than the consumption ΔC_H relative to Hohmann's solution.

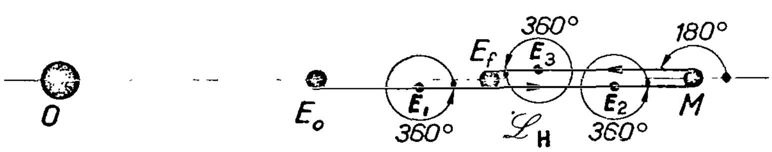


Fig. 10. Multi-impulsional Transfer.

II,4.3.2.1. Particular Case of Two Maximal Acceleration Arcs for a Transfer Between Close Circular Orbits.

The integral $\dot{I}$ in this case becomes:

$$\dot{I} = \frac{1}{2}\int_{(\mathscr{L})} e^2(1+\cos^2 v)\frac{\dot{d}e}{\cos v} = \rho^3\left(\frac{5}{2}\varphi - 2\sin\varphi - \frac{\sin 2\varphi}{4}\right) \qquad (23)$$

where ψ is the angle at the center of the festoons $\left(\varphi = \frac{S}{2\rho} = \frac{\Delta a}{4\gamma_{max}}\right)$ (Figure 12).

The relative separation of the characteristic frequencies between the solution with two thrust arcs and Hohmann's bi-impulsional solution is, therefore, for the same transfer:

$$\frac{\Delta C - \Delta C_H}{\Delta C_H} = \frac{3\Delta a^2}{4}\left[\frac{1}{12} - \left(\frac{\gamma_{max}}{\Delta a}\right)^3 (20\,\varphi - 16\sin\varphi - 2\sin 2\varphi)\right](1 + \text{order } \varepsilon)$$

a formula already found by McIntyre [42] which is also written:

$$\frac{\Delta C - \Delta C_H}{\Delta C_H} = \frac{\Delta a^2}{128}\left(8 - 3\,\frac{10\varphi - 8\sin\varphi - \sin 2\varphi}{\varphi^3}\right)(1 + \text{order } \varepsilon) \tag{24}$$

and if ψ is small:

$$\frac{\Delta C - \Delta C_H}{\Delta C_H} = \frac{\Delta a^2 \varphi^2}{128}\left[1 + \text{order } \max(\varepsilon, \varphi^2)\right] \tag{25}$$

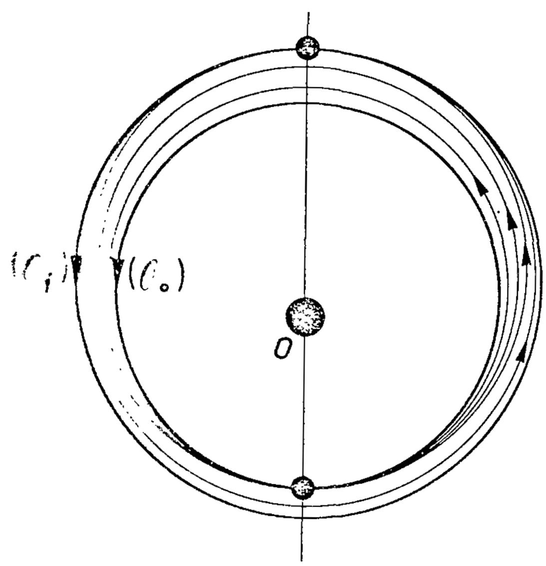


Fig. 11. Multi-impulsional Transfer.

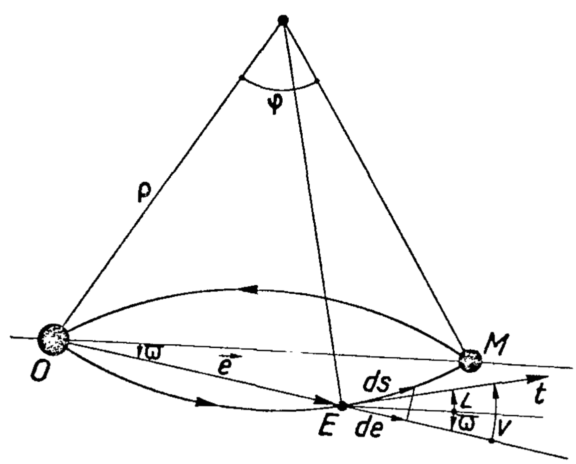


Fig.12.Two Maximal Acceleration Arcs.

The supplement of characteristic velocity referring to Hohmann's solution is therefore of the order $\varepsilon^3\psi^2$, and this is true no matter what ψ is, i.e. the importance of the two thrust arcs (Figure 13).

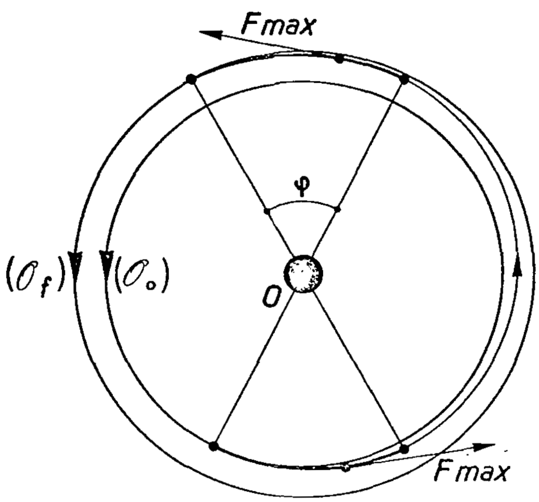


Fig. 13. Two Maximal Acceleration Arcs.

II,4.3.2.2. Any Number of Arcs. /125

The condition $\cos v \equiv 1$ being characteristic of Hohmann's solution achieving the optimum, it is necessary to expect that the optimal solution with limited acceleration will be such that $\cos v \simeq 1$, at least if the maximal acceleration is large enough for the impaired transfer duration, which we shall presume.

This is equivalent to an extension of the thrust arcs around the impulsional points corresponding to maximal "efficiency". This is really the case for transfer between close circles using two propulsed arcs analyzed in the preceding paragraph. The integral $\mathring{I}$, is written again, for $\sin v \simeq 0$:

$$\dot{I} = \frac{1}{2}\int_{(\mathscr{E})} e^2(1+\cos^2 v)\frac{de}{\cos v} = \int_{(\mathscr{E})} e^2\left(1+\frac{\sin^4 v}{8} + \text{order}\,\sin^6 v\right)|de| \tag{26}$$

or by integration:

$$\dot{I} = -\frac{(e_0)^3+(e_f)^3}{3} + \frac{2}{3}e_M^3 + \text{order}\; e_M^3 \sin^4 v \tag{27}$$

where e_M is the maximal eccentricity found during transfer.

Let us suppose that the perigee festoons (that is to say corresponding to the line $\widehat{E_0M}$) have fixed lengths S_k (Figure 14). The lengths C_k of the corresponding cords are fixed. $\dot{I}$ is maximal for maximal e_M, i.e. when the cords are aligned on the straight line OE_0E_f without the perigee line (L_p) showing any retrogression.

Now let us suppose that under this alignment hypothesis only the total length:

$$S_p = \sum_{k=1}^{n_p} S_k = \rho \sum_{k=1}^{n_p} \varphi_k \tag{28}$$

of the perigee festoons is fixed. Then:

$$e_M - e_o = \sum_{k=1}^{n_p} c_k = p \sum_{k=1}^{n_p} \left(\varphi_k - \frac{\varphi_k^3}{24} + \text{order } \varphi_k^5\right) \tag{29}$$

is maximum when $\sum_{k=1}^{n_p} \varphi_k^3$ is minimum, that is to say for equal perigee festoons ($\psi_k = \psi_p$; $k = 1.2 \ldots, n_p$)

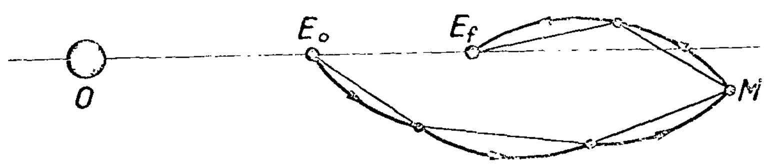


Fig. 14. Succession of Maximal Acceleration Arcs and Ballistic Arcs.

Analogous reasoning would demonstrate the equality of the apogee festoons ($\psi_k = \psi_A$; $k = n_p + 1, \ldots, n$) when the total length S_A of the apogee festoons is fixed.

If at the present time only the total length $S = S_p + S_A$ of the path (L) is fixed, the perigee festoons and the apogee festoons must still be respectively equal to each other (Figures 15 and 16). In fact, if the perigee festoons, as an example, were not equal to each other, it would be possible to obtain the same value of e_M with a perigee length $S'_p < S_p$, therefore with a total length $S' = S'_p + S_A < S = S_p + S_A$ and the solution would not be optimal.

A more precise calculation [taking into consideration the term $e_M^3 \sin^4 v$ neglected in (27)] shows that the alignment of the cords is only approximately /126 realized (the cords of the festoons form angles of the order of ψ^3 with the straight line OE_0E_f) and that the relative differences between the length of the perigee or apogee festoons are of the order of ψ^2. The extra precision is useless in this study if $\psi^6 \leq$ order ε.

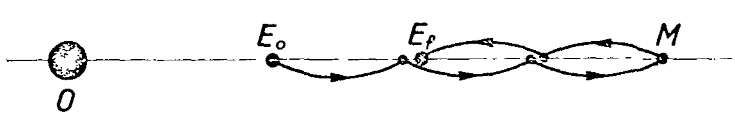


Fig. 15. Optimal Solution.

The numbers n_p, n_A and the respective magnitudes of the perigee and apogee festoons still have to be determined, i.e. to make:

$$J_1 = \frac{e_M - \frac{1}{2}\left(e_0 + e_f\right)}{\rho} = n_P \sin \frac{\varphi_P}{2} + n_A \sin \frac{\varphi_A}{2} \qquad (30)$$

maximal, with the constraints:

$$n_P + n_A = n = E\left(N + \frac{3}{2}\right) = \text{ fixed integer} \qquad (31)$$

n = number of thrust arcs, E = integral part.

N = number of revolutions authorized by the transfer period.

(As long as $N << \frac{8}{\Delta a^2}$, which is practically always the case, this is equivalent to fixing the duration Δt of the transfer or the number of revolutions N).

$$n_P \ \varphi_P + n_A \ \varphi_A = \frac{S}{\rho} = \Lambda \text{ fixed} \qquad (32)$$

$$2 n_P \sin \frac{\varphi_P}{2} - 2 n_A \sin \frac{\varphi_A}{2} = \frac{e_f - e_0}{\rho} = \frac{\Delta e}{\rho} = \Delta \text{ fixed} \qquad (33)$$

$$n_P, n_A = \text{ integers.} \qquad (34)$$

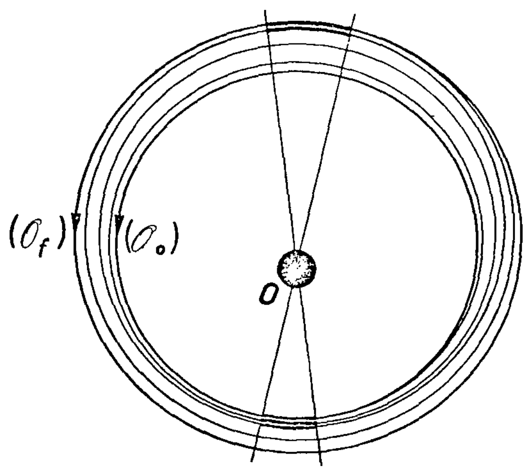


Fig. 16. Optimal Solution.

In a first approximation we shall replace $\sin \frac{\psi}{2}$ by $\frac{\psi}{2}$ in (33): we then derive: /127

$$2 n_P \ \varphi_P = \Lambda + \Delta = 2F \geqslant 0 \qquad (35)$$

$$2 n_A \ \varphi_A = \Lambda - \Delta = 2G \geqslant 0 \qquad (36)$$

$$J = 24\left(\Lambda - 2J_1\right) = \frac{F^3}{n_P^2} + \frac{G^3}{n_A^2} = J_P + J_A \cdot \qquad (37)$$

If the condition (34) is not considered, J is minimum for $n_p = n_p^*$, $n_A = n_A^*$, not necessarily integers, and ψ_A, $= \psi_P = \psi^*$ (Figure 17). In consideration of (34), it is necessary to choose between the values $n_p = E(n_p^*)$ and $n_p = E(n_p^* + 1)$ with $n_A = n - n_p$, which makes J minimal. In general there is no ambiguity. Let us note that if $\Delta e = 0$, $F = G = \frac{\Lambda}{2}$, curves J_p and J_A are symmetrical, and therefore: $n_p^* = n_A^* = \frac{n}{2}$. If $n = 2r$ is an even number, there is only one optimal solution offering r thrust arcs at perigee and r thrust arcs at perigee.

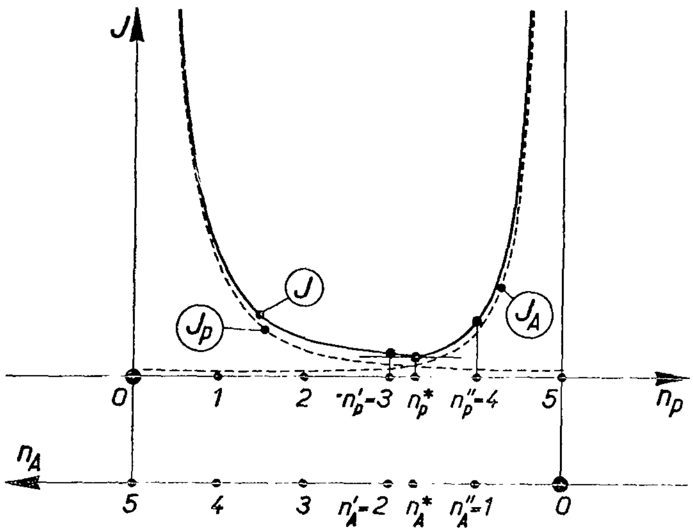


Fig. 17.

If $n = 2r + 1$ is an odd number, ambiguity exists, because J assumes the same value for the solution where the supplementary thrust arc takes place at perigee and the solution where it takes place at apogee.

The ambiguity can only be removed by a calculation of higher orders.

Then we get $n_p = r + 1$ and $n_A = r$.

II,4.3.2.3. Thrust Arc Periods of Duration.

First of all we shall suppose that the propulsion system has limited acceleration $(0 \leq \gamma \leq \gamma_{max})$.

Let $\delta t_{pk} = \frac{\delta C_{pk}}{\gamma_{max}}$, the thrust arc period at perigee number k.

δC_{pk} is the corresponding consumed characteristic velocity:

$$\delta C_{pk} = -\delta B_{pk}\left(1 + \text{order}\,\varepsilon^2\right) \tag{38}$$

now:

$$B = P^{-1/2}(1+e)^{-1/2}(1-e^2)^{1/4} = P^{-1/2}\left(1 - \frac{e}{2} + \frac{e^2}{8} + \text{order } e^3\right) \tag{39}$$

where P is the distance from the center of attraction to perigee. /128

Therefore, for a thrust arc to the perigee:

$$\delta B = -\frac{1}{2}P^{-3/2}\delta P(1+e)^{-1/2}(1-e^2)^{1/4} + P^{-1/2}\left(\frac{\delta e}{2} + \frac{e\delta e}{4}\ \text{order}\ {}^2\delta e\right). \tag{40}$$

In order to simplify the calculations, we shall presume that the thrust arcs are short. The perturbation formula of the distance from perigee then shows that:

$$\frac{\delta P}{\delta C} = \text{order } \psi^2 \quad (\psi = \text{angle at the festoon center}) \tag{41}$$

that is to say that the perigee shifts very little, therefore:

$$\delta B = P_o^{-1/2}\left(-\frac{\delta e}{2} + \frac{e\,\delta e}{4}\right)\left\{1 + \text{order } max\left[\varepsilon^2, \left(k - \frac{1}{2}\right)\varphi^2\right]\right\} \tag{42}$$

and

$$\delta t_{PK} = \frac{P_o^{-1/2}}{\gamma_{max}}\left(\frac{\delta e_{PK}}{2} - \frac{e_{PK}\,\delta e_{PK}}{4}\right)\left\{1 + \text{order } max\left[\varepsilon^2, \left(k - \frac{1}{2}\right)\varphi^2\right]\right\} \tag{43}$$

All variations δe_{pk} are equal to $\delta e_p = \frac{e_M - e_0}{n_p}$, the length of the perigee festoon cords. e_{pk} is the mean value of e for perigee arc number k.

$$\delta t_{PK} = \frac{P_o^{-1/2}(e_M - e_o)}{2 n_P \gamma_{max}}\left(1 - \frac{e \text{ mean on arc } k}{2}\right)\left\{1 + \text{order } max\left[\varepsilon^2, \left(k - \frac{1}{2}\right)\varphi^2\right]\right\} =$$

$$\frac{P_o^{-1/2}(e_M - e_o)}{2 n_P \gamma_{max}}\left[1 - \frac{e_o}{2} - \frac{\left(k - \frac{1}{2}\right)}{2 n_P}(e_M - e_o)\right]\left\{1 + \text{order } max\left[\varepsilon^2, \left(k - \frac{1}{2}\right)\varphi^2\right]\right\} \tag{44}$$

There would be a similar demonstration for the apogee arcs:

$$\delta t_{AK} = \frac{A_f^{-1/2}(e_M - e_f)}{2 n_A \gamma_{max}}\left(1 + \frac{e \text{ mean on arc } k}{2}\right)\left\{1 + \text{order } max\left[\varepsilon^2, \left(n - k + \frac{1}{2}\right)\varphi^2\right]\right\} =$$

$$\frac{A_f^{-1/2}(e_M - e_f)}{2 n_A \gamma_{max}}\left[1 + \frac{e_f}{2} + \frac{\left(n - k + \frac{1}{2}\right)(e_M - e_f)}{2 n_A}\right]\left\{1 + \text{order } max\left[\varepsilon^2, \left(n - k + \frac{1}{2}\right)\varphi^2\right]\right\}. \tag{45}$$

Therefore the thrust arcs periods decrease in arithmetical progression, in the ratio $-\frac{P_o^{-1/2}}{\gamma_{max}}\left(\frac{e_M - e_o}{2 n_P}\right)^2$ for the perigee arcs and $-\frac{A_f^{-1/2}}{\gamma_{max}}\left(\frac{e_M - e_f}{2 n_A}\right)^2$ for the apogee arcs.

It is easy to verify that the ratio:

$$r = \frac{\text{sum of the apogee arcs periods}}{\text{sum of the perigee arc periods}} = \frac{\sum_{k=n_p+1}^{n} \delta t_{Ak}}{\sum_{k=1}^{n_p} \delta t_{Pk}} \tag{46}$$

in the case of transfer between circular orbits ($e_0 = e_f = 0$) is equal to: /129

$$r = 1 - \frac{\Delta a}{4} \tag{47}$$

i.e. the ratio $\frac{\Delta V_A}{\Delta V_P}$ of Hohmann impulses.

If it is a propulsion system with limited thrust ($F \leq F_{max}$) and no longer with limited acceleration, the progressive relaxation of the mobile permits a supplementary shortening of the thrust arc period. The period of arc number k is no longer δt_k but $\delta t'_k = \delta t_k + \delta(\delta t_k)$ with:

$$\frac{\delta(\delta t_k)}{\delta t_k} = \frac{m_k - m_0}{m_0} = -\frac{C_k}{W} = \frac{B_k - B_0}{W} = \begin{cases} -\dfrac{P^{-1/2}}{W}\dfrac{e_{Pk}-e_0}{2} \simeq -\dfrac{e_{Pk}-e_0}{2W} \\ -\dfrac{P^{-1/2}}{W}\dfrac{e_M - e_0}{2} + \dfrac{A^{-1/2}}{W}\dfrac{e_{Ak}-e_M}{2} \simeq \\ \dfrac{e_k + e_0 - 2e_M}{2W} \end{cases} \tag{48}$$

where W is the ejection velocity and e_k is the mean value of e for the festoon k.

Whence:

$$\delta t'_{Pk} = \frac{P_0^{-1/2}(e_M - e_0)}{2n_P\,\gamma_{max}}\left[1 - \frac{e_0}{2} - \frac{\left(k - \frac{1}{2}\right)}{2n_P}(e_M - e_0)\left(1 + \frac{1}{W}\right)\right]\left\{1 + \text{order}\,max\left[\varepsilon^2, \left(k - \frac{1}{2}\right)\varphi^2\right]\right\} \tag{49}$$

and

$$\delta t'_{Ak} = \frac{A_f^{-1/2}(e_M - e_f)}{2n_A\,\gamma_{max}}\left[1 + \frac{e_f}{2}\,\frac{\left(n - k + \frac{1}{2}\right)}{2n_A}(e_M - e_f)\left(1 + \frac{1}{W}\right) + \frac{e_f + e_0 - 2e_M}{2W}\right]$$
$$\times\left\{1 + \text{order}\ max\left[\varepsilon^2, \left(n - k + \frac{1}{2}\right)\varphi^2\right]\right\} \tag{50}$$

Let us note that (49) and (50) are respectively reduced to (44) and (45) when W is infinite.

II,4.3.2.4. Angles at the Center of the Thrust Arcs.

The angle δL_{pk}, swept by the vector radius during the propulsed phase number k at perigee, is connected to the period δt_{pk} of this phase by:

$$\left(\frac{\delta L}{\delta t}\right)_{Pk} = \left(\frac{h}{r^2}\right)_{Pk} = P_0^{-3/2}\left(1+\frac{e_{Pk}}{2}\right)\left\{1 + \text{order: } max\left[\varepsilon^2, \left(k-\frac{1}{2}\right)\varphi^2\right]\right\} \tag{51}$$

(h = kinetic moment),
from which, by using equation (49):

$$\delta L_{Pk} = \frac{P_0^{-2}(e_M - e_0)}{2n_P\,\gamma_{max}}\left(1-\frac{e_{Pk}-e_0}{2W}\right)\left\{1 + \text{order } max\left[\varepsilon^2, \left(k-\frac{1}{2}\right)\varphi^2\right]\right\} \tag{52}$$

Likewise, for the thrust arcs at apogee: /130

$$\delta L_{Ak} = \frac{A_f^{-2}(e_M - e_f)}{2n_A\,\gamma_{max}}\left(1+\frac{e_{Ak}+e_0-2e_M}{2W}\right)\left\{1 + \text{order } max\left[\varepsilon^2, \left(n-k+\frac{1}{2}\right)\varphi^2\right]\right\} \tag{53}$$

The angles δL_k therefore decrease with arithmetical progression in the ratio:

$$\frac{P_0^{-2}}{\gamma_{max}W}\left(\frac{e_M-e_0}{2n_P}\right)^2 \quad \text{for the perigee arcs and}$$

$$-\frac{A_f^{-2}}{2\gamma_{max}W}\left(\frac{e_M-e_f}{2n_A}\right)^2 \quad \text{for the apogee arcs.}$$

In the case of a propulsion system with limited acceleration, it is sufficient to make $W = \infty$ in the preceding formulae. The angles at the center of the thrust arcs at perigee and at apogee are respectively equal among each other, at approximately the order max $(\varepsilon^2\psi,\ n\,\psi^3)$.

For example, for a transfer with a large even number of thrust arcs $n = 2r$, between circular orbits, if δt_1 is the duration of the first propulsed arc and δL_1 is the corresponding angle to the center, the duration of the last perigee arc is $\delta t_r = \delta t_1\ (1 - \frac{\Delta a}{4})$ and the corresponding angle $\delta L_r = \delta L_1$. The period of the first apogee arc is $\delta t_{r+1} \simeq \delta t_r$ and the corresponding angle $\delta L_{r+1} = \delta L_1\ (1 - 2\Delta a)$. Finally the duration of the last apogee arc is: $\delta t_n = \delta t_1\ (1 - \frac{\Delta a}{2})$ and the corresponding angle is $\delta L_n = \delta L_{r+1}$.

II,4.3.3. Loss in Ratio to the Hohmann Solution in the Circular Case.

Formula (24) enumerates the loss due to the extension of the thrust on two maximal acceleration arcs for a transfer between close circles, in a ratio to the bi-impulsional solution of Hohmann.

We propose to establish a more general formula, although less precise, in the case of any number of arcs.

The characteristic velocity of an optimal solution between circles is given by:

$$\Delta C = -\Delta B - \frac{e_M^3}{3} + \text{order } \max(\varepsilon^4, \varepsilon^3 \varphi^4) \tag{54}$$

where

$$e_M = 2 n_P \rho \sin\frac{\varphi_P}{2} = 2 n_A \rho \sin\frac{\varphi_A}{2} \tag{55}$$

with

$$\rho = 2\gamma_{max} = \frac{2\Delta C}{n_P \varphi_P + n_A \varphi_A} = \frac{\Delta a}{n_P \varphi_P + n_A \varphi_A} \tag{56}$$

whence

$$e_M = \frac{\Delta a}{2}\left(1 + \frac{1}{24}\ \frac{n_P \varphi_P^3 + n_A \varphi_A^3}{n_P \varphi_P + n_A \varphi_A} + \text{order } \varphi^3\right). \tag{57}$$

Now, (55) shows that

$$n_P \varphi_P = n_A \varphi_A (1 + \text{order } \varphi^2)$$

therefore

$$\Delta C = -\Delta B - \frac{\Delta a^3}{32}\left(1 + \frac{\varphi_P^2 + \varphi_A^2}{16} + \text{order } \varphi^4\right) \tag{58}$$

whence /131

$$\frac{\Delta C - \Delta C_H}{\Delta C} = \frac{\Delta a^2(\varphi_P^2 + \varphi_A^2)}{128}\left[1 + \text{order } \max(\varepsilon, \varphi^2)\right] \tag{59}$$

for n = 2, we again find (25).

II,4.4. NEAR-OPTIMAL SOLUTIONS.

Let us take up equation (10) again:

$$-\frac{dC}{dB} = 1 + \frac{1}{2}\left(\varphi - \frac{e}{2}\sin v\right)^2 - \frac{3e^2}{8}\left(1 + \cos^2 v\right) + \text{order } M^3. \tag{60}$$

For a Hohmann solution between circular orbits, (with a possible fragmentation of impulses) $\psi \equiv 0$ and $\sin v \equiv 0$, therefore

$$-\frac{dC}{dB} = 1 - \frac{3}{4}e^2 + \text{order } \mathcal{E}^3 \tag{61}$$

and by integration:

$$-\frac{\Delta C}{\Delta B} = 1 - \frac{e_M^2}{4} + \text{order } \mathcal{E}^3 = 1 - \frac{\Delta a^2}{16} + \text{order } \mathcal{E}^3. \tag{62}$$

Now suppose that the orientation of the thrust is only restricted to

$$\left|\varphi - \frac{e}{2}\sin v\right| = \left|\varphi_B\right| \leqslant \frac{\sqrt{3}}{2} e \sqrt{1 + \cos^2 v} \tag{63}$$

where ψ_B is the angle of the thrust with the *bisectrix* of the local horizontal and the tangent.

Then the most unfavorable solution (i.e. leading to maximal consumption ΔC, ΔB being given) corresponds to $e \equiv 0$ (spiral) and:

$$-\frac{dC}{dB} = 1 + \text{order } M^3 \tag{64}$$

or, by integration:

$$-\frac{\Delta C}{\Delta B} = 1 + \text{order } M^3 \tag{65}$$

Therefore, for every solution where the thrust orientation satisfies (63), consumption ΔC differs little from the Hohmann consumption.

$$\Delta C_H \leqslant \Delta C \leqslant \Delta C_H\left(1 + \frac{\Delta a^2}{16}\right). \tag{66}$$

This is particularly true if the angle $|\psi_B|$ of the thrust with the bisectrix defined above is less than $\min \frac{\sqrt{3}}{2} e \sqrt{1+\cos^2 v} = \sqrt{\frac{3}{2}}\, e = \sqrt{\frac{3}{2}}\, x$ (maximum angle between the tangent and the horizontal obtained at the peak of the minor axis).

II,4.5. CONCLUSION.

Plane transfers of limited but sufficient duration between non-intersecting, direct, coaxial, near-circular, close orbits are achieved in an economic way by using Hohmann's bi-impulsional solution, if the propulsive thrust is not limited.

In the opposite case, they consist of a succession of maximal thrust arcs /132
with a slowly decreasing period, first at perigee and then at apogee. The optimal thrust is practically applied according to the bisectrix of the angle formed by the local horizontal and the tangent to the trajectory.

The angles at the center of the perigee and apogee arcs are respectively equal among each other if the ejection velocity is large compared with the orbiting velocity, and otherwise they slowly decrease.

The difference between the characteristic velocity of such a solution and the characteristic velocity of Hohmann's solution for the same transfer is of the third order in respect to the size of the transfer, even for relatively long thrust arcs. *This explains the degeneracy determined in the linearized study of such solutions.*

II,5. OPTIMAL IMPULSE TRANSFERS BETWEEN CLOSE NEAR-CIRCULAR ORBITS, COPLANAR OR NON-COPLANAR.

II,5.1. INTRODUCTION.

The analytical study of optimal transfers between near-circular ($e \simeq 0$), Keplerian, near orbits, coplanar or not, is particularly important because the orbits found in practice often have a weak eccentricity and the problem of slightly modifying them is frequently met.

Such a transfer can be defined (Figure 1) by the vector $\vec{\vec{\Delta j}}$, rotation of the plane of the orbit, the variation $\vec{\vec{\Delta e}}_{plan}$ of the projection of the perigee vector $\vec{e}$ onto the plane of the nominal orbit (*O*) and finally the relative "dilatation" $\Delta a/a$ of the semi-major axis.

If the eccentricity e of the nominal orbit is of the order $\leq \varepsilon$ (= the /133
size of the transfer), i.e. for a transfer between *near-circles* proper, we have seen that it is the same as supposing it to be zero in the linearized study.

The angle δ between the major axis $\vec{\vec{Ox}}$ of the nominal orbit (*O*) and the node line (support for $\vec{\vec{\Delta j}}$) does not then occur in the calculations and the transfer can be defined by *four scalar parameters* instead of five. These four parameters are:

rotation Δj, the component $\Delta e_{//}$ of $\vec{\vec{\Delta e}}$ parallel to $\vec{\vec{\Delta j}}$, the component $\Delta e_{\perp}$ of $\vec{\vec{\Delta e}}$ perpendicular to $\vec{\vec{\Delta j}}$ and finally "relative dilatation" $\Delta a/a$ of the semi-major axis.

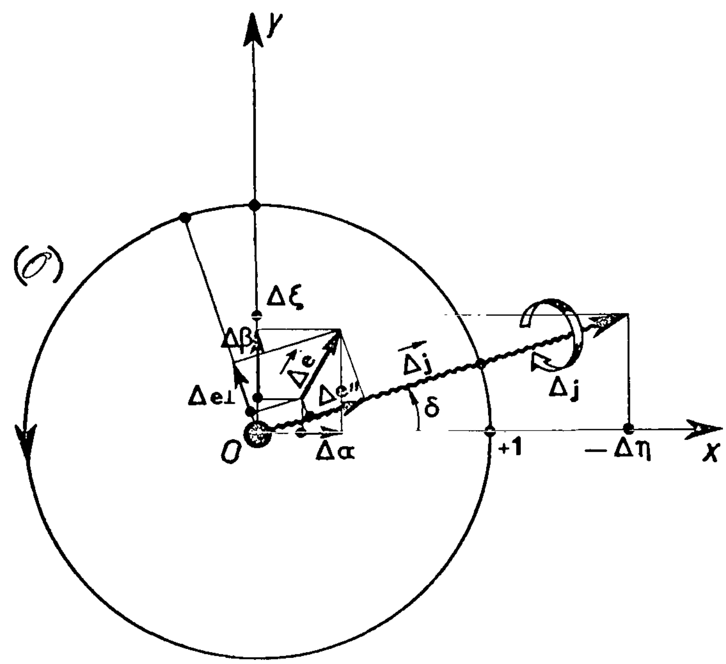


Fig. 1. Notations.

On the other hand, if the eccentricity of the nominal orbit is of the order $\geq \varepsilon$, even though small compared with unity, i.e. in the case of transfer between *elliptical orbits with weak eccentricity*, angle δ occurs in the calculations. Then the transfer is defined by five parameters.

These two cases will be envisaged in succession.

(Except in degenerated cases) we shall only be concerned with the *impulsional* solutions with refer to propulsion systems of the type (S_1). These solutions appear when

$$\lambda = \frac{\varepsilon}{\Delta L \; F_{max}} \ll 1$$

where ΔL is the transfer angle and F_{max} is the maximal thrust. If the propulsive is capable of furnishing finite impulses (F_{max} >> local attraction), we shall also suppose that the transfer angle ΔL is large enough to make it possible to use the solution obtained in the calculations by making the hypothesis of the indifferent transfer angle. Stated otherwise the transfer arc $\widehat{M_0 M_f}$ should contain the points where the impulses obtained in this hypothesis (one revolution is enough in every case) are applied.

II,5.2. TRANSFERS BETWEEN NEAR-CIRCULAR ORBITS ($e \leq$ order $\varepsilon \ll 1$).

This problem has been studied parallelly by Gobetz, Washington and Edelbaum [48 - 49].

II,5.2.1. Optimal Thrust.

II,5.2.1.1. Symmetries.

Since linearization is made around a circular nominal orbit (O), the choice of the axis of reference $\vec{Ox}$ is arbitrary. Since this choice was made best in the study under consideration (Figure 1), the transfer is defined by the variations $-\Delta\eta$ and $\Delta\xi$, characterizing the rotation Δj, $\Delta\alpha$ and $\Delta\beta$ characterizing the variation $\vec{\Delta e}$ and finally Δa.

A thrust acceleration $\vec{\gamma}$ in the direction $\vec{D}(X,Y,Z)$ at point M of direct ascent L during the time dt, produces the variation:

$$\left\{\begin{aligned} d\xi &= \sin L\ Z\ \gamma\ dt \\ d\eta &= -\cos L\ Z\ \gamma\ dt \\ da &= 2\,Y\,\gamma\ dt \\ d\alpha &= (X \sin L + 2\,Y \cos L)\ \gamma\ dt \\ d\beta &= (-X \cos L + 2\,Y \sin L)\ \gamma\ dt \end{aligned}\right. \tag{1}$$

In this study of symmetries, let us take the axis of reference $\vec{Ox}$ according to the node line, support for the vector $\vec{\Delta j}$. $d\alpha$ and $d\beta$ then represent respectively $de_{//}$ and $de_{\perp}$.

Instead of applying the thrust acceleration (γ; X, Y, Z) at point of direct ascent L during the time dt, if it is applied at the diametrically opposite point, direct ascent $L + \pi$, da is unchanged, while $\vec{dj}$ and $\vec{de}$ change direction; therefore the total result is that Δa is unchanged, while $\vec{\Delta j}$ changes direction, but $\Delta e_{//}$ and $\Delta e_{\perp}$ are unchanged because $\vec{\Delta e}$ also changes direction. /134

Likewise, if instead of applying thrust acceleration (γ; X, Y, Z) at point L for the time dt, we apply acceleration (γ; -X, -Y, -Z) to the point $L + \pi$, da changes sign , while $\vec{dj}$ and $\vec{de}$ are unchanged; therefore the total result is that Δa changes sign, while $\vec{\Delta j}$ and $\vec{\Delta e}$, therefore Δj, $\Delta e_{//}$ and $\Delta e_{\perp}$ are unchanged.

If instead of applying the thrust acceleration (γ; X, Y, Z) for time dt at point L, we apply acceleration (γ; -X, Y, -Z) at point $\pi - L$ (symmetrical to the first by relationship to $\vec{Oy}$), $d\eta$, da and $d\beta$ are unchanged, while $d\xi$ and $de_{//}$ change sign; therefore the total result is that $\vec{\Delta j}$ is unchanged (because $\vec{\Delta\eta} = \Sigma\vec{d\eta} = -\vec{\Delta j}$ is unchanged and $\vec{\Delta\xi} = \Sigma\vec{d\xi} = 0$ remains zero), Δa and $\Delta e_{\perp}$ are unchanged while $\Delta e_{//}$ changes sign.

Finally, if instead of applying thrust acceleration (γ; X, Y, Z) for time dt at point L, we apply acceleration (γ; -X, Y, Z) at point -L (symmetrical to the first by the relationship to $\overrightarrow{Ox}$), $d\eta$, da, $de_{//}$ are unchanged, while $d\xi$ and $de_{\perp}$ change signs; therefore the total result is that $\overrightarrow{\Delta j}$ is unchanged (because $\overrightarrow{\Delta\eta} = \Sigma\overrightarrow{d\eta} = -\overrightarrow{\Delta j}$ is unchanged and $\overrightarrow{\Delta\xi} = \Sigma\overrightarrow{d\xi} = 0$ remains zero), Δa and $\Delta e_{//}$ are unchanged, while $\Delta e_{\perp}$ changes sign.

These results are summed up in the table below.

$\Delta e_{//}, \Delta e_{\perp}, \Delta j, \Delta a$	L	X	Y	Z
$-\Delta e_{//}$	$\pi - L$	$-X$	Y	$-Z$
$-\Delta e_{\perp}$	$-L$	$-X$	Y	Z
$-\Delta j$	$\pi + L$	X	Y	Z
$-\Delta a$	$\pi + L$	$-X$	$-Y$	$-Z$

Thus, changing the sign of only one of the four transfer parameters can be realized, *at the equal cost of* $\Delta C = \int_{to}^{tf} \gamma dt$, by modifying the position L from the point of thrust application and from the direction $\vec{D}$ of thrust.

Only the *moduli* $|\Delta e_{//}|$, $|\Delta e_{\perp}|$, $|\Delta j|$, $|\Delta a|$ of the variations thus take place in the results about the type of solution to be used and about the accessible domain.

II,5.2.1.2. Efficiency Curve.

The not-necessarily zero components of the kinematic adjoint $\vec{p}$ are p_{ξ}, p_{η}, p_a, p_{α}, p_{β}.

In § I,3.2.4. we saw that, in the case of circular nominal orbit, the efficiency curve (P) is an ellipse (Figure I,3 - 6), a section of the elliptical cylinder (σ) [of generatrices parallel to the axis $\overrightarrow{MZ}$ and of base ellipse (P) centered in ω (X = 0, Y = $2p_a$, Z = 0) of major axis $\overrightarrow{MY}$ ($a_{(P)} = 2p_e$, $b_{(P)} = p_e$)] and of plane (π) passing through ω.

Impulsional solutions of an indifferent period correspond to an ellipse (P) *inside* the sphere (Σ) with center M and radius 1 and *touching* this sphere in one or two (or an infinity of) points. Thus each point of contact furnishes a point of application M(L) and a direction $\vec{D} = \overrightarrow{MP}$ of optimal thrust, this thrust being able to be applied once (one impulse) or several times during successive resolutions (fragmentation of impulse).

A necessary condition to assure the above-mentioned contact is for ellipse (P) to be entirely within (Σ) or, if need be, tangent to (Σ).

The sphere (S) enclosing the major circle (C) of (P) as the principal circle (Figure 2) is itself entirely inside (Σ) or, if need be, tangent to (Σ), or even coincident with (Σ). The points of contact of *P* corresponding to the impulses on (Σ) are therefore outside (S) or, if necessary, on (S) and therefore, in the dihedron, with crest $\overrightarrow{\mathcal{M}Y}$, with faces ($\pi^+$) and ($\pi^-$) containing the circles (c^+) and (c^-), sections of the sphere (S) through the cylinder (σ). /135

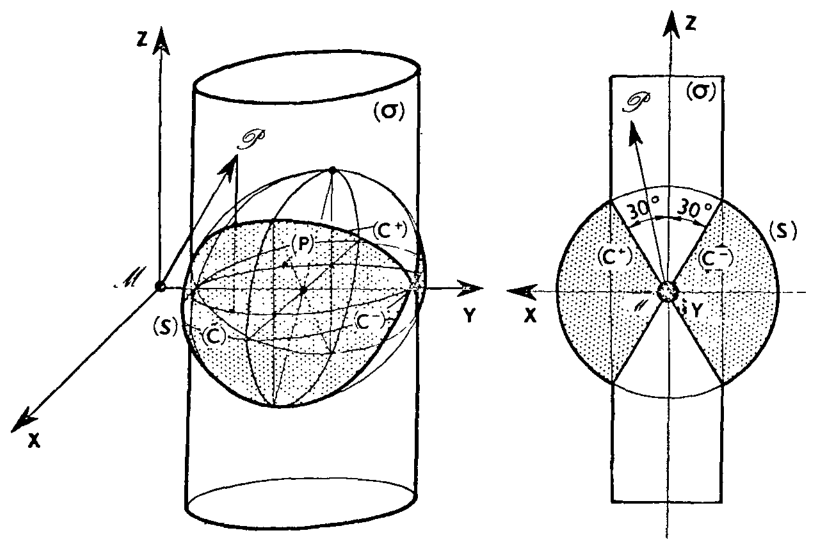


Fig. 2. Useful Spatial Angle.

An optimal impulse, directed according to $\vec{D} = \overrightarrow{\mathcal{M}\mathcal{P}}$ is therefore contained in one of the two dihedrons of 60° formed by the planes (π^+) and (π^-) forming angles of ±30° with the local horizontal plane X = 0. We come across a property of the "useful spatial angle" studied by C. Marchal [12].

In § 4 of Appendix 13 it is shown that every optimal mono-impulsional solution (a single impulse per revolution) can be considered in two different fashions as the limit of a bi-impulsional solution when one of the two impulses tends toward zero. Therefore we can limit ourselves to the multi-impulsional case.

II,5.2.1.3. The Four Types of Linearized Solutions.
(The demonstrations appear in Appendix 13).

According to the expression:

$$\max\left(\Delta a^2, \Delta e_{//}^2, \Delta e_{//}^2 + \Delta e_{\perp}^2 + \frac{2}{\sqrt{3}}|\Delta j||\Delta e_{\perp}| - \Delta j^2\right) = \begin{cases} \Delta a^2 & I \\ \Delta e_{//}^2 & II \\ \Delta e_{//}^2 + \Delta e_{\perp}^2 + \frac{2}{\sqrt{3}}|\Delta j||\Delta e_{\perp}| - \Delta j^2 & III \end{cases} \quad (3)$$

the optimal solution belongs to one of the following three fundamental types (Figure 3):

Type I (bi-impulsional).

If ω is not in M ($p_a \neq 0$), the ellipse (P) centered in ω on $\overrightarrow{MY}$ cannot be bi-tangent to (Σ) at two points P' and P'' unless its plane (π) passes through $\overrightarrow{MY}$ ($\vec{p}_e$ and $\vec{p}_z$ colinear).

The two impulses $I'\Delta C$ and $I''\Delta C$ are applied at points $M'(L')$ and $M''(L'' = -L')$, symmetrical in relation to the axis $\overrightarrow{Ox}$ forming angle $-\delta$ with the node line, the support of the rotation vector $\overrightarrow{\Delta j}$. /137

$tg\delta$ is the root of the largest modulus of the second degree equation:

$$\Delta e_{\perp} \Delta e_{//}^2 tg^2\delta - (\Delta e_{//}^2 - \Delta e_{\perp}^2 - \Delta a^2 - \Delta j^2)\Delta e_{//} tg\,\delta - \Delta e_{\perp}(\Delta e_{//}^2 - \Delta a^2) = 0 \quad (4)$$

and

$$-1 \leqslant \cos L' = \frac{\Delta a}{\Delta e_{//}} \cos\delta \leqslant +1 \quad (5)$$

$$-1 \leqslant I' - I'' = \frac{\Delta e_{//}}{\Delta a} \frac{\sin L'}{\sin\delta} \leqslant +1 \quad (6)$$

$$X' = -X'' = \frac{(\Delta e_{//}^2 - \Delta a^2)\cos\delta - \Delta e_{//}\Delta e_{\perp}\sin\delta}{\Delta e_{//}\Delta C \sin L'} \quad (7)$$

$$Y' = Y'' = \frac{\Delta a}{2\Delta C} \quad (8)$$

$$Z' = -Z'' = \frac{|\Delta j|}{\Delta C}\frac{\sin\delta}{\sin L'}. \quad (9)$$

(This presumes that the impulse $I'\Delta C$ corresponds to $\sin L' > 0$).

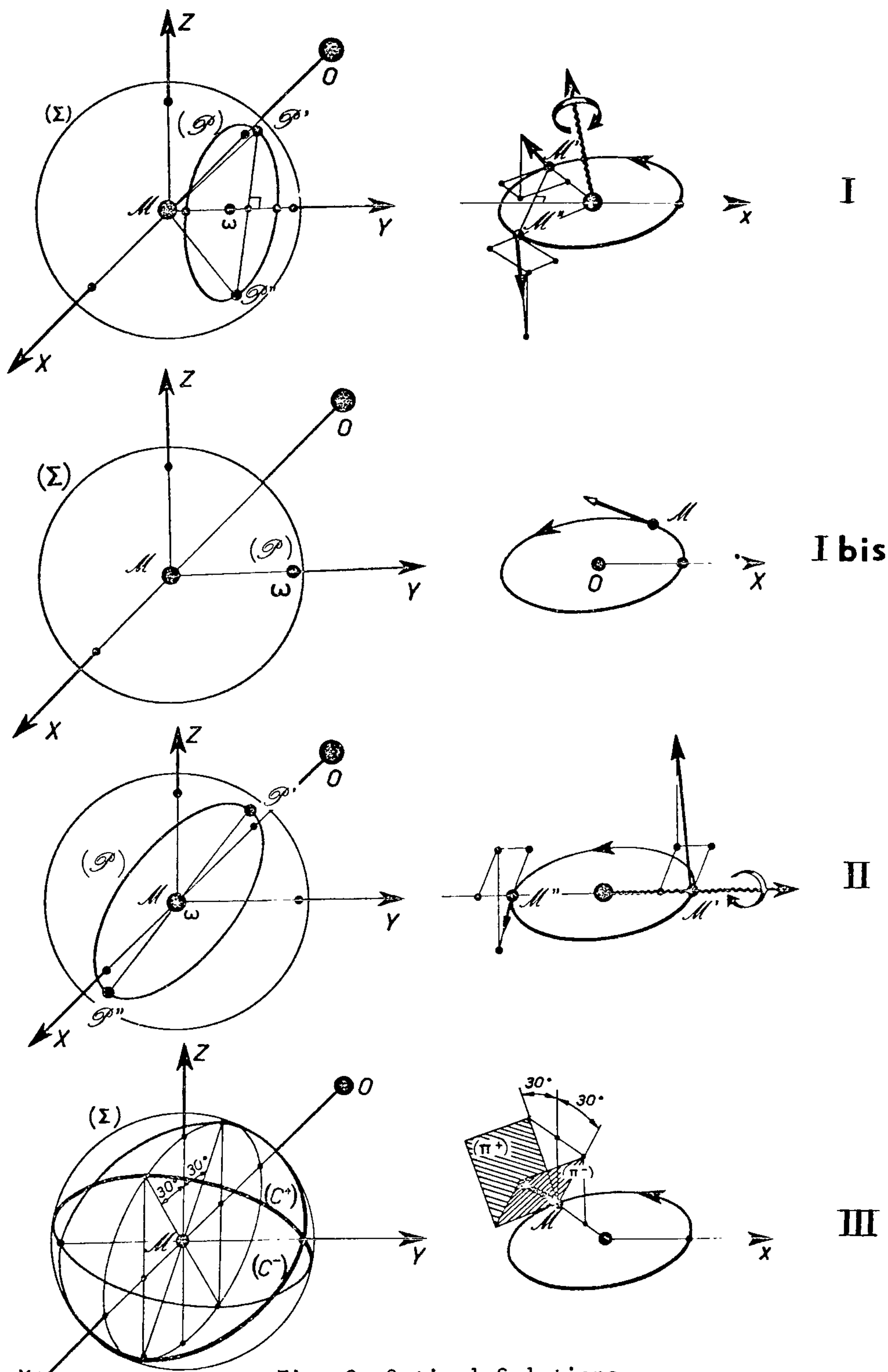


Fig. 3. Optimal Solutions.

If at the same time $p_e = 0$ & $p_a = \frac{\varepsilon a}{2}$ ($\varepsilon a = \pm 1$), the ellipse (P) is reduced at its center ω on sphere (Σ). We obtain *the singular plane case of type I bis* already pointed out in § I,3.2.6.

The thrust is tangential. There is degeneracy of the linearized solution, for in the first order every point of the orbit can be suitable for applying thrust. Taking up again the direct reasoning from §II,3.3.2.5, we see that any degenerate solution at all is obtained by distributing on the circle (O) with a total fictitious "mass" $\Delta C = |\Delta a|/2$, with a linear density $F(L) \leq F_{max}$, so that the center of gravity of this distribution is in G, such that:

$$\overrightarrow{OG} = \frac{\overrightarrow{\Delta e}}{\Delta a}. \tag{10}$$

If $|\Delta e| \leq |\Delta a|$, this can always be achieved, *particularly* with the help of two impulses [G inside of (O)] or with a single impulse [G on (O)] (Figure 4).

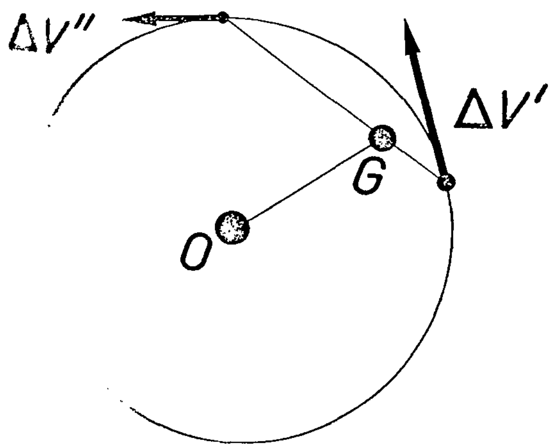


Fig. 4. Type I bis. Bi-Impulsional Solutions.

In § II,5.3.1. we shall see that such a first order degeneracy does not appear if linearization is made around a nominal orbit of weak eccentricity ($\varepsilon << e << 1$) and that the optimal linearized solutions are then mono- or bi-, or sometimes even tri-, impulsional.

Type II (bi-impulsional, nodal). /138

If ω is in $M(p_a = 0)$ and if ellipse (P) is not bi-tangent to (Σ) ($p_e \neq 1/2$), the points of contact P' and P'' of (P) and of (Σ) correspond to two points M' and M'' diametrically opposed on (O) ($L'' = L' + \pi$).

The two impulses $I'\Delta C$ and $I''\Delta C$ are applied at the nodes with:

$$-1 \leq I' - I'' = \frac{\Delta a}{\Delta e_{//}} \leq +1 \tag{11}$$

$$X' = -X'' = -\frac{\Delta e_{\perp}}{\Delta C} \tag{12}$$

$$Y' = -Y'' = \frac{\Delta e_{//}}{2\Delta C} \tag{13}$$

In the mobile axes, the two optimal impulse directions are opposed. In their fixed axes they are symmetrical in relation to the plane of the orbit.

[continued]

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$$\left. Z' = -Z'' = \frac{|\Delta j|}{\Delta C} \right\} \quad (14)$$

Type III (singular, tri-dimensional).

If ω is in $M(p_a = 0)$ and if ellipse (P) is bi-tangent to (Σ) ($p_e = 1/2$), ellipse (P) must necessarily coincide with one of the major circles (C^+) or (C^-) of the sphere (Σ), intersection of the sphere (Σ) and of the cylinder (σ) situated in the planes (π^+) or (π^-) going through $\overrightarrow{\overrightarrow{MY}}$ and forming angles of ±30° with the local horizontal plane X = 0.

Then the vectors $\overrightarrow{\overrightarrow{p_e}}$ and $\overrightarrow{\overrightarrow{p_z}}$ are colinear and such that:

$$\vec{P_z} = \pm \sqrt{3}\, \vec{P_e}. \quad (15)$$

We again find the singular solution of type III already pointed out in § I,3.2.6. There is first order degeneracy, since every point of the orbit can theoretically be suitable for applying thrust. Optimal thrust is contained in one of the planes (π^+) or (π^-) and its direction is really determined as a function of the position.

Taking the vector $\overrightarrow{\overrightarrow{p_e}}$ as the origin axis $\overrightarrow{\overrightarrow{Ox}}$:

$$\vec{D} \begin{cases} X = \dfrac{\sin L}{2} \\ Y = \cos L \\ Z = \pm \dfrac{\sqrt{3}}{2} \sin L \end{cases} \quad (16)$$

The application of the thrust F(L), on arc dL surrounding point M(L) of orbit (O) and in the optimal direction $\vec{D}$, produces the variations:

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$$\begin{cases} d\xi = \pm \dfrac{\sqrt{3}}{2} \sin^2 L \; F(L)\, dL \\ d\eta = \mp \dfrac{\sqrt{3}}{2} \sin L \cos L \; F(L)\, dL \\ da = 2 \cos L \; F(L)\, dL \end{cases} \quad (17)$$

[cont. on next page]

$$\begin{cases} d\alpha = \left(2 - \frac{3}{2}\sin^2 L\right) F(L)\, dL \\ d\beta = \frac{3}{2}\sin L \cos L \, F(L)\, dL \\ dC = F(L)\, dL \end{cases} \tag{17}$$

[cont. from previous page]

We notice that:

$$\vec{de} \pm \sqrt{3}\,\vec{dZ} = \vec{de} \mp \sqrt{3}\,\vec{dj_1} = 2\vec{dC}. \tag{18}$$

Or, by integration:

$$\vec{\Delta e} \pm \sqrt{3}\,\vec{\Delta Z} = \vec{\Delta e} \mp \sqrt{3}\,\vec{\Delta j_1} = 2\vec{\Delta C} \tag{19}$$

where $\vec{\Delta j}_1$ is deduced from $\vec{\Delta j}$ by the rotation of axis $\vec{Oz}$ and of angle $+\pi/2$ and where the consumption vector $\vec{\Delta C}$ has ΔC as its length and is directed according to $\vec{Ox}$ (i.e. $\vec{p_e}$), which fixed the direction of reference $\vec{Ox}$ as a function of transfer elements.

The transfers of type III corresponding to a fixed *vector* $\vec{\Delta C}$, therefore particularly to a characteristic fixed velocity $\Delta C = |\vec{\Delta C}|$, can thus be defined by the datum of the rotation vector $\vec{\Delta j}$ and of Δa [vector $\vec{\Delta e}$ is then deduced from $\vec{\Delta j}$ and $\vec{\Delta C}$ by (19)], that is to say by the datum of a point G in the axes

$$x_1 = \mp \frac{2}{\sqrt{3}}\Delta\eta\,, \; y_1 = \pm \frac{2}{\sqrt{3}}\Delta\xi\,, \; z_1 = \frac{\Delta a}{2}\,. \qquad \text{(Figure 5).}$$

By applying impulse ΔC to point $M(L)$ in the optimal direction $\vec{D}$, we reach a point $G \equiv V$ situated on Viviani's Window (V) with parametric equations:

$$(V) \begin{cases} x_1 = \Delta C \sin L \cos L \\ y_1 = \Delta C \sin^2 L \\ z_1 = \Delta C \cos L \end{cases} \tag{20}$$

With the same characteristic velocity ΔC it is possible to reach any point G inside the volume $(\mathcal{V})$ limited by the smallest convex contour surrounding (V) by composing an impulse of type V, so that their sum is equal to ΔC and that

G is the center of gravity of the points V treated as fictitious "masses" equal to the impulse magnitude in M. It is even possible to envisage, in the extreme, a modulation of the thrust [distribution of the "mass" ΔC on the orbit (O) with linear density $F(L) \leq F_{max}$]. Thus the center σ of circle (C) can be reached by applying a constant thrust F in the optimal direction (Figure 12).

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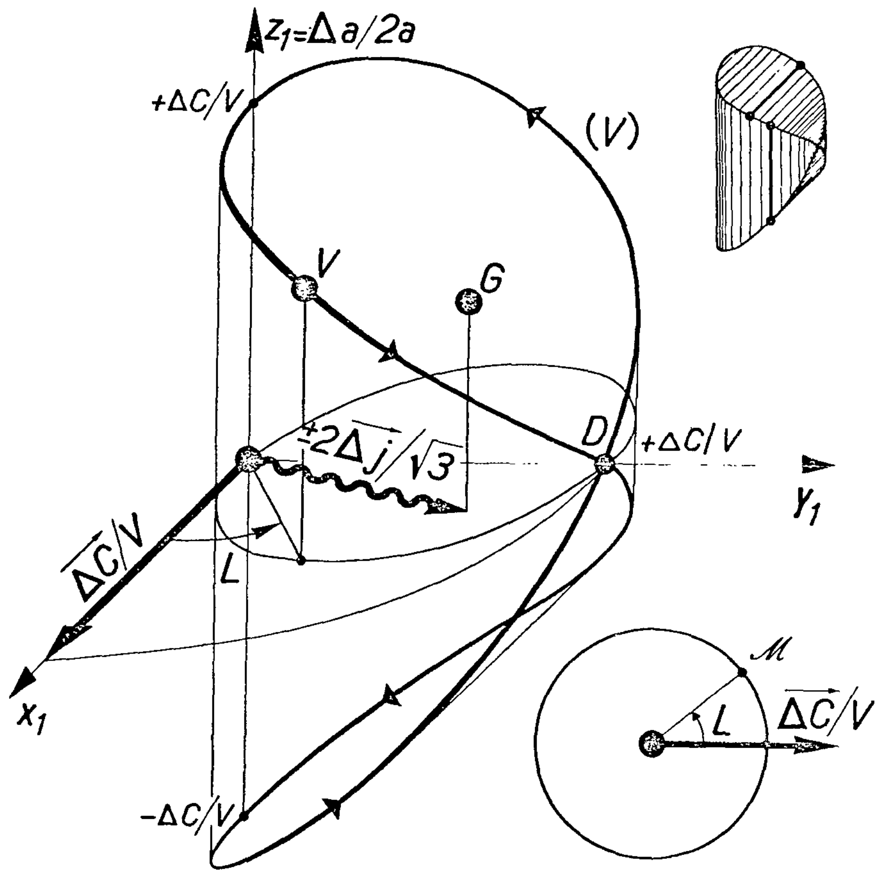

Fig. 5. Type III. Viviani's Window.

If a point of the right cylinder (C_I) with parabolic base (P) or of the right cylinder (C_{II}) with a circular base (C) makes up the frontier surface of the domain, (V) is obtained with the aid of two impulses. These non-degenerate solutions assure transition with solutions of type I or II.

Then the question comes of knowing whether, among all the degenerate optimal solutions referred to a point G inside the volume (V) limited by (C_I)

and (C_{II}), there do or do not exist bi-impulsional solutions. The answer is affirmative: *every point* G *can be obtained by a bi-impulsional solution, and that in two different ways*.

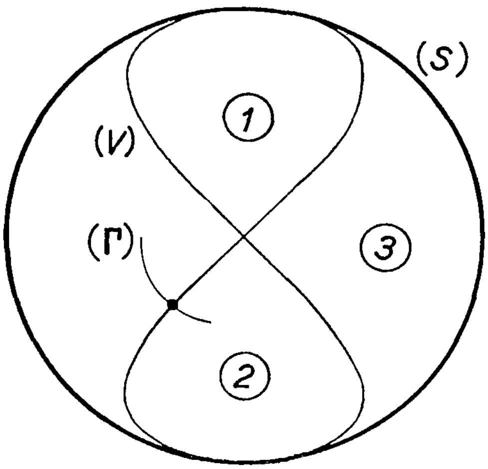

Fig. 6.

In fact, on the sphere (S) on which it is plotted, Viviani's Window (V) separates the regions 1, 2 and 3 (Figure 6). Let G be a point inside volume (*V*). The cone of peak G lying on (V) cuts across sphere (S) according to a curve (Γ). Let V be a point of intersection of (V) and of (Γ). The straight line GV cuts across, by definition of (Γ), Viviani's Window in V', point V' itself being on (Γ) since V is on (V). Therefore the segment VGV' corresponds to a bi-impulsional solution, with the impulses being applied in L(V) and L'(V').

Therefore, there are as many bi- /141
impulsional solutions relative to point G as there are pairs VV'. Therefore, the number of these solutions is equal to the number of points of intersection of (V) and (Γ) divided by 2.

Let us then consider the section of the figure through the plane $y_1 = C^{te}$ passing through G. This plane cuts the sphere (S) according to a circle (C') and the Window (V) according to four points V_1, V_2, V_3, V_4 (Figure 7).

The diagonals V_1V_3 and V_2V_4 of rectangle $V_1V_2V_3V_4$ separate the superior (s), inferior (i) right lateral (d) and left (g) regions.

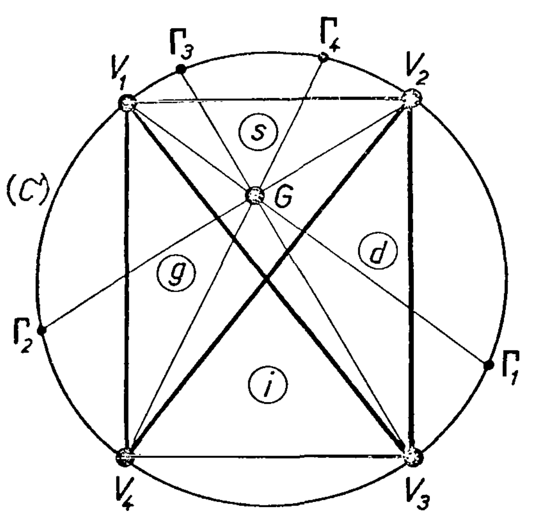

Fig. 7

It is evident that points Γ_1, Γ_2, Γ_3, Γ_4 are part of (Γ). If G is in region (s) (Figure 7) Γ_3 and Γ_4 are both in region (1), while Γ_1 and Γ_2 are in region (3). Since (Γ) is a continuous curve, there are always *at least* two intersection points between (Γ) and (V), i.e. a bi-impulsional solution relating to G. The same result is found if G is in region (i).

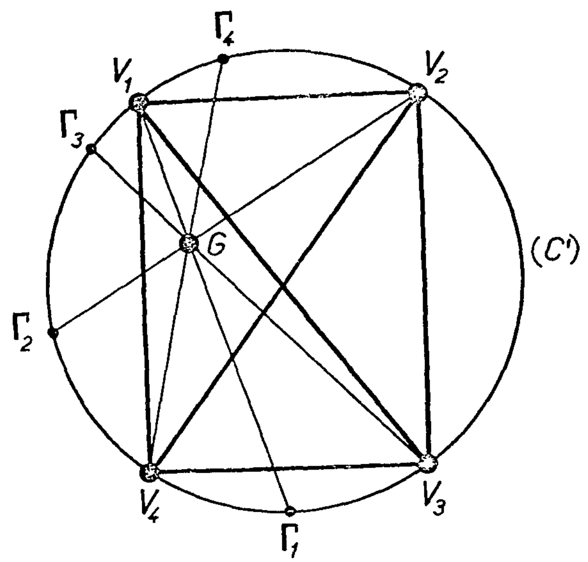

Fig. 8.

If G is in region (d) or (s) (Figure 8), it is likewise shown that there are *at least* four intersection points between (Γ) and (V), i.e. two bi-impulsional solutions relative to G.

Let us now show that in all these cases there are two,and only two,bi-impulsional solutions. For this it is enough to demonstrate that there exists at least one point G inside (*V*) which enjoys this property and that by continuity this property is extended to the totality of the volume (*V*). For this let us take G in the center σ of circle (C) (Figure 9).

Let VV' be a cord of (V) /142
passing through σ. The circular projection of (V) on the plane $z_1 = 0$ shows that σ must be the center of VV'. Therefore let (V') be Viviani's Window, symmetrical to (V) in relationship to σ. (V) and (V') have in common only the points V_1, V_2, V_3 and V_4, as the projection on to plane $x_1 = 0$ shows. Therefore there are only two bi-impulsional solutions corresponding to the pairs V_1V_3 and V_2V_4.

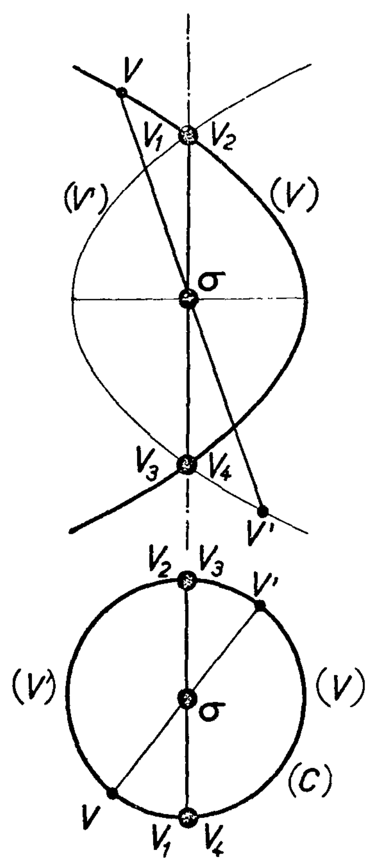

Fig. 9.

Starting with point σ, let us suppose that we find a frontier inside (*V*) separating this two-solution zone from a zone with one or three solutions. The curves (V) and (Γ) are then tangent, therefore bi-tangent, and the tangents to the contact points are co-planar [in the plane tangent to the cone of peak G lying on (V)]. This case can only occur for (V) if G is on one of the cylinders (C_I) or (C_{II}) limiting (*V*). In that case the two bi-impulsional solutions coincide in one.

The property mentioned has therefore been demonstrated (Figure 10).

The longitudes L_1, L_2, L_3, L_4, of the impulse points corresponding to the two bi-impulsional solutions V_1V_3 and V_2V_4 passing through

G are connected by a relationship independent of the position of G obtained by writing that points V_1, V_2, V_3, and V_4 of Viviani's Window (V) are in the same plane (defined by V_1V_3 and V_2V_4 intersecting at G). This relationship is:

$$t_1t_2t_3 + t_2t_3t_4 + t_1t_3t_4 + t_1t_2t_4 + t_1 + t_2 + t_3 + t_4 = 0 \tag{21}$$

where $t = \mathrm{tg}\,\frac{L}{2}$.

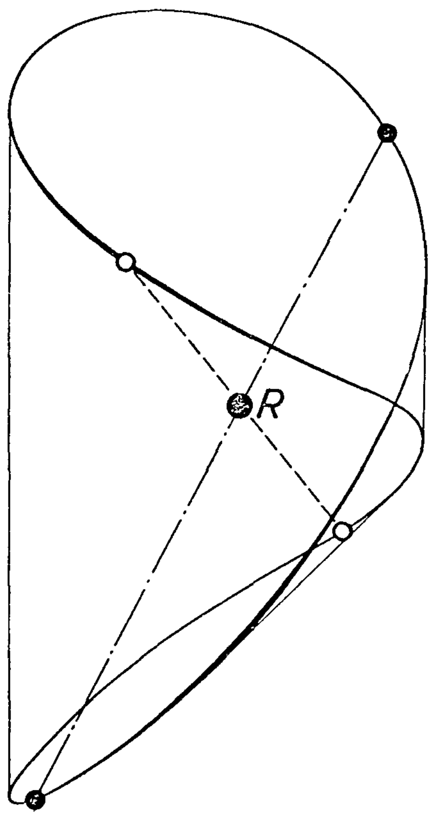

Fig. 10. Type III.
Bi-impulsional Solutions.

NOTE: In § II,5.3.2. we shall see that the first order degeneracy found for solutions of type III disappears if the linearization is made around a nominal orbit with weak eccentricity ($\varepsilon \ll e \ll 1$). Then the optimal solutions are mono-, bi- or tri-impulsional.

II,5.2.2. Accessible Domain.

With a given characteristic velocity ΔC, it is possible to reach all the points situated in a certain hypervolume called "accessible domain" in four-dimensional space of variations $\Delta e_{//}$, $\Delta e_{\perp}$, Δj and Δa.

We shall represent this domain by its sections $\Delta a = C^{te}$ in the three-dimensional space $\Delta e_{//}$, $\Delta e_{\perp}$, Δj (Figure 11) reduced to the octant $\Delta e_{//} > 0$, $\Delta e_{\perp} > 0$, $\Delta j > 0$ by the considerations of symmetry made at the beginning of the study.

The economic transfers evidently correspond to the points of the surface of the accessible domain.

The surface of the fourth degree CFNMC (Figure 11b) of equation:

$$\left(-\Delta e_{//}^2 + \frac{3}{4}\Delta a^2 + \Delta C^2\right)\left(\Delta e_{\perp}^2 + \Delta j^2 + \frac{\Delta a^2}{4} - \Delta C^2\right) + \Delta e_{\perp}^2\left(\Delta e_{//}^2 - \Delta a^2\right) = 0 \tag{22}$$

corresponds to solutions of type I.

The part FNF_2F of the ellipsoid of revolution /146

$$\frac{\Delta e_{//}^2}{4} + \Delta e_{\perp}^2 + \Delta j^2 = \Delta C^2 \tag{23}$$

of the axis of revolution $\overrightarrow{OF_2}$ corresponds to solutions of type II.

/143

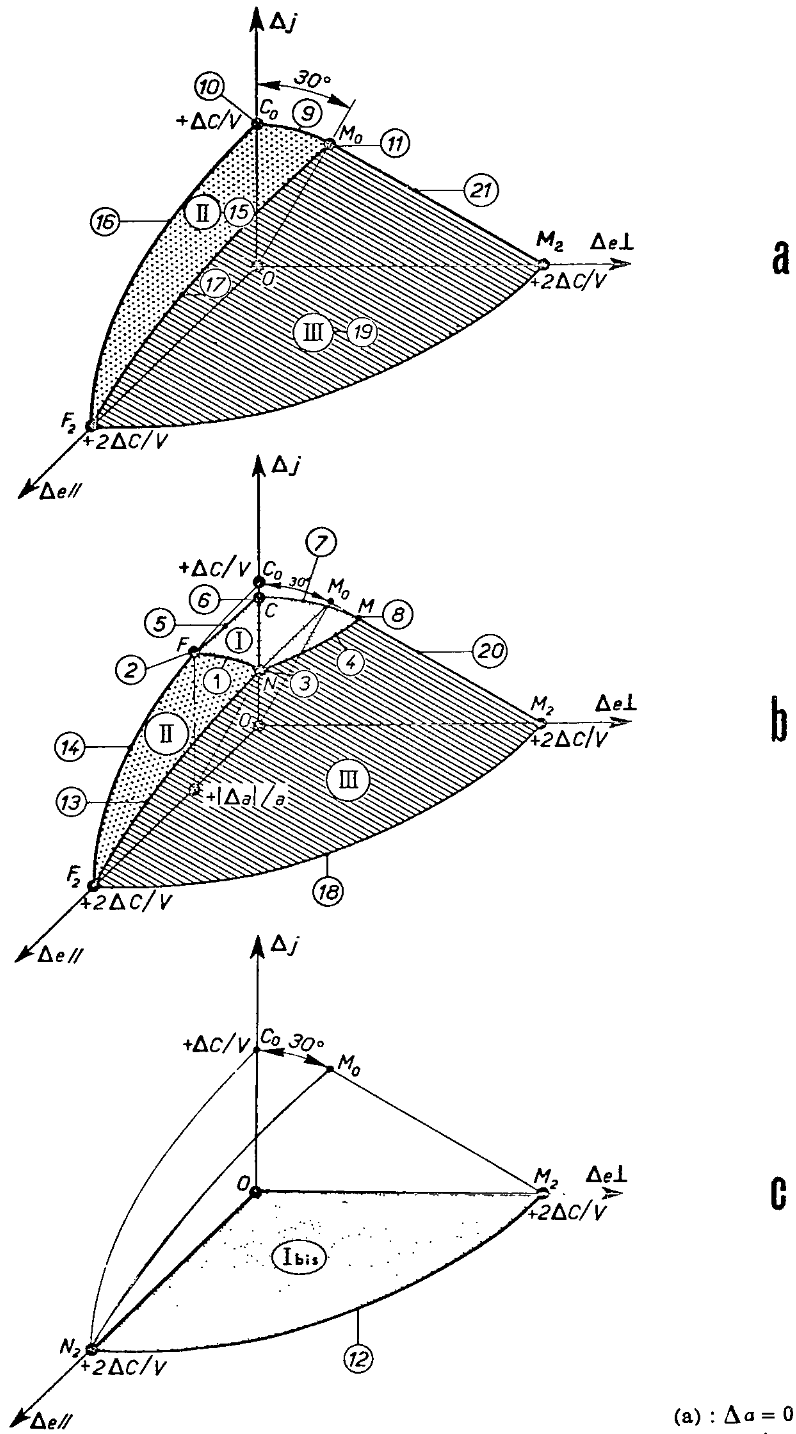

Fig. 11. Accessible Domain.

(a) : $\Delta a = 0$

(b) : $0 < |\Delta a| < 2\,\Delta C$

(c) : $|\Delta a| = 2\,\Delta C$.

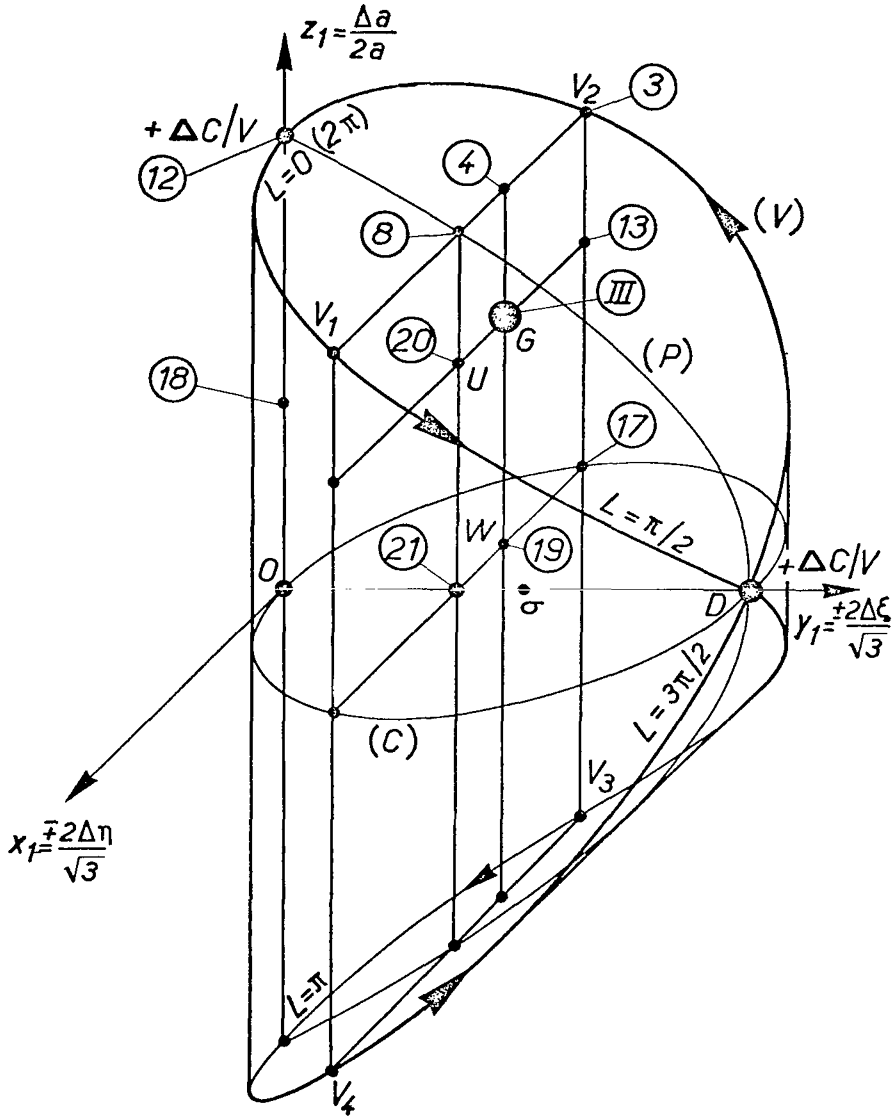

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Fig. 12. Solutions of the Type III.

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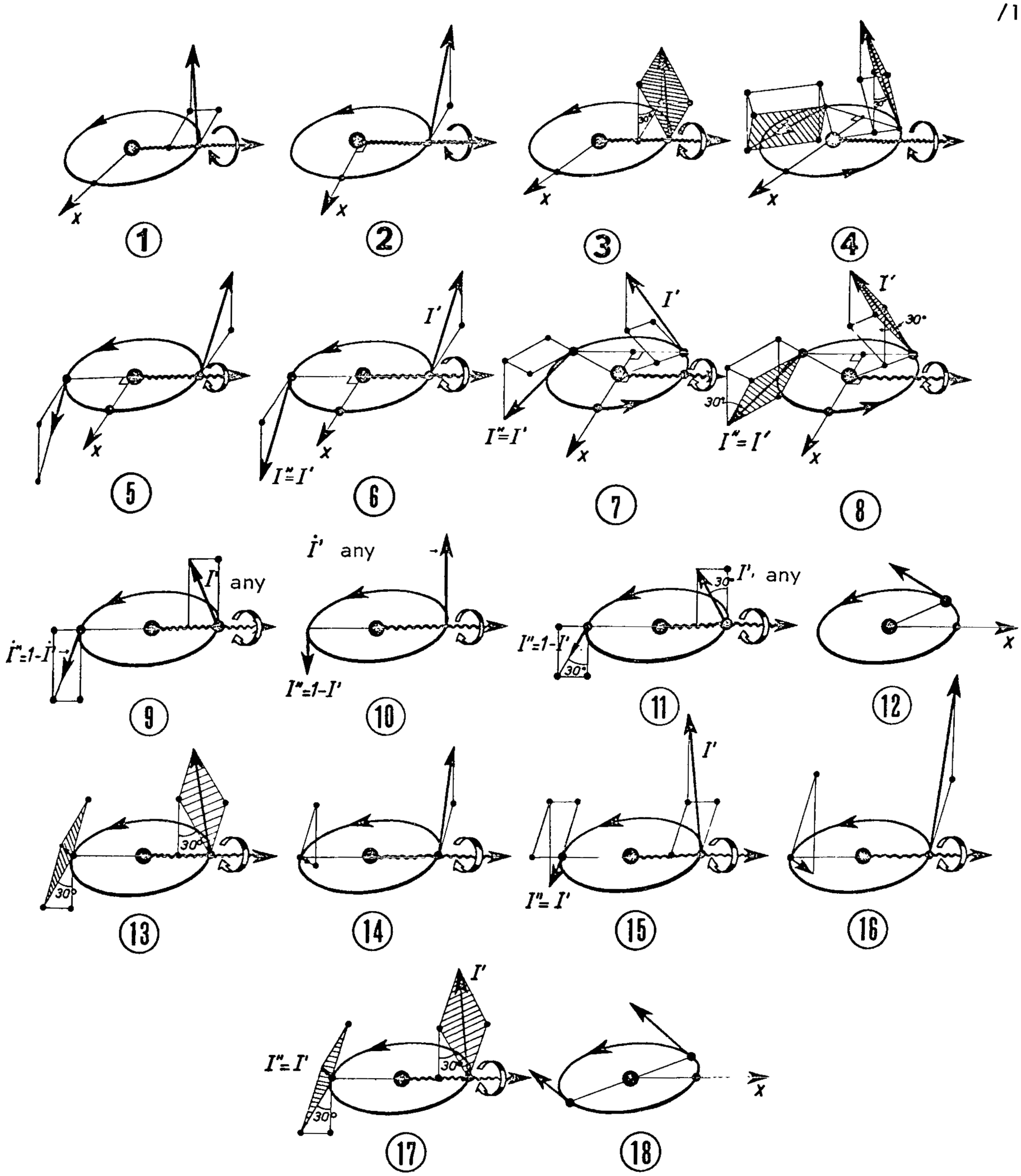

Fig. 13. Particular Solutions.

Finally, $F_2NMM_2F_2$ of the elliptical cylinder:

$$\frac{\Delta e_{/\!/}^2}{4} + \left(\frac{|\Delta e_\perp|}{2} + \frac{\sqrt{3}}{2} |\Delta j|\right)^2 = \Delta C^2 \tag{24}$$

lying on the ellipsoid (II) corresponds to solutions of type III.

The vectorial relationsip (19) is shown in Figure 14.

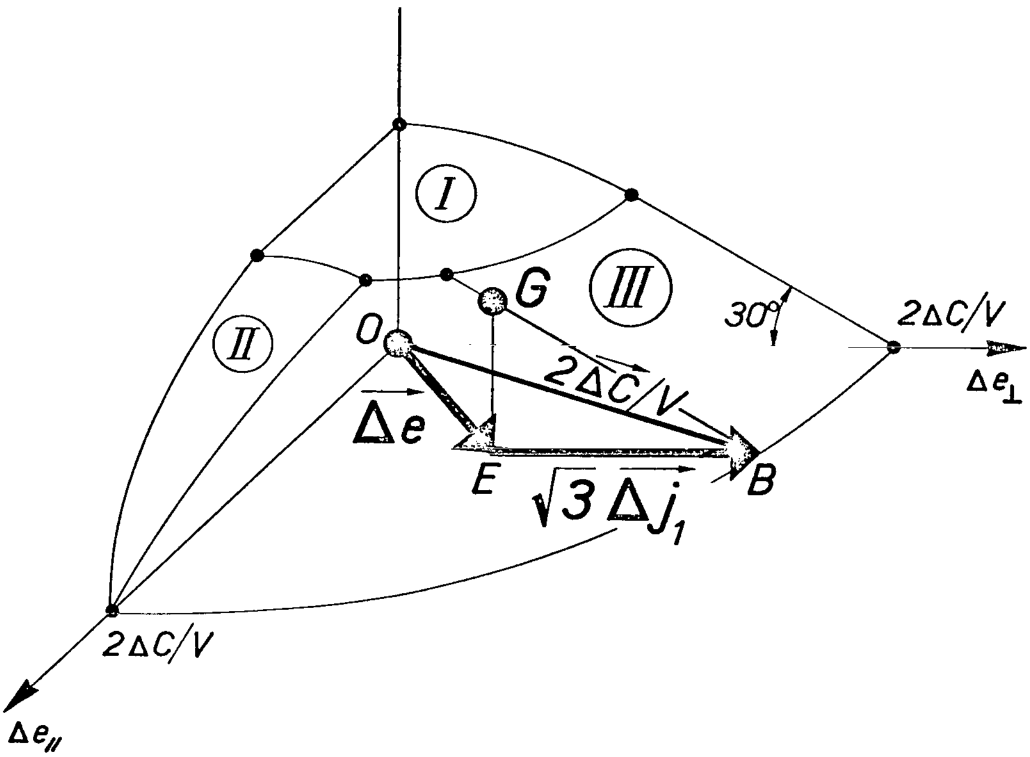

Fig. 14. Type III. Consumption Vector $\vec{\Delta C}$.

When the relative dilatation of the semi-major axis varies from 0 to 2ΔC, part corresponding to solutions of type I increases from the arc of circle $\widehat{C_0M_0}$ (Figure 11a) to the circular sector ON_2M_2O (Figure 11c - type I bis). The surfaces relative to solutions of type II and III remain fixed, and only their surface diminishes (Figures 15, 16, 17, 18).

The solutions corresponding to the numbers shown in Figures 11 and 12 are described in Figure 13.

In order to fix the orders of magnitude, let us give two deliberately very simple examples corresponding to two particular points of the "accessible domain" (Figure 19).

Let there be a circular, circumterrestrial orbit at 500 km altitude. /149

An impulse ΔC of 10m/s perpendicular to the plane of the orbit produces a lateral separation of 9 km at the end of one quarter revolution.

The same impulse, applied tangentially forward, produces an altitude gain of 36 km accompanied by a longitudinal lag of 85 km at the end of a half revolution.

II,5.2.3. Conclusion.

Optimal transfers between close, near-circular orbits, co-planar or not, with eccentricity of the same order of magnitude as the size of the transfer, can always be realized in the first order with only two impulses (exceptionally one impulse). In certain degenerate cases, greater liberty is left for the choice of the optimal solution.

Only a higher order study permits a determination of the exact number of impulses necessary to achieve optimal transfer. Breakwell [16] has shown that in the vicinity of the arc $\overset{\frown}{C_0M_0}$ of Figure 11a, there exists a zone corresponding to tri-impulsional solutions. Second order study of the solutions of types I bis and III should also be within reach.

In the following paragraph we shall satisfy ourselves with *the linearized study* of these solutions in the case of elliptical orbits with weak eccentricity and no longer of near circles. /150

II,5.3. TRANSFERS BETWEEN ELLIPTICAL ORBITS OF SLIGHT ECCENTRICITY ($\varepsilon << e << 1$).

The study of impulsional solutions close to the solutions of types I and II found in § II,5.2. in the case where linearization was made around a circle ($e = 0$) would lead here to results qualitatively similar to those already found and differing from the latter only by quantities of the order of e.

On the other hand, the study of impulsional solutions close to the solutions of types I bis and III (degenerate in the first in the case where linearization was made around a circle) is of interest because it leads to qualitatively very different results (disappearance of the first order degeneracy). Therefore we shall limit ourselves to this study.

The choice of axes is indicated in Figure 1. $\vec{\vec{Ox}}$ is directed this time toward the perigee P of the nominal orbit (O).

Here a supplementary transfer parameter is introduced: it is angle δ between the major axis $\vec{\vec{Ox}}$ of the nominal orbit and node line $\vec{\vec{\Delta j}}$.

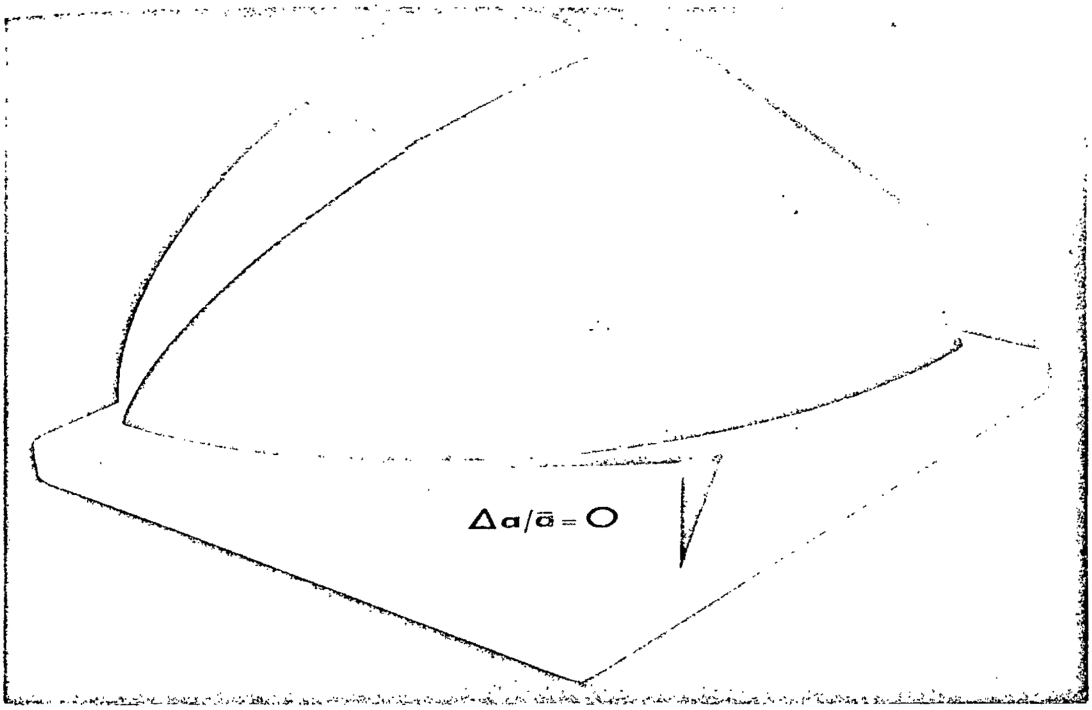

Fig. 15. Accessible Domain $\Delta a = 0$.

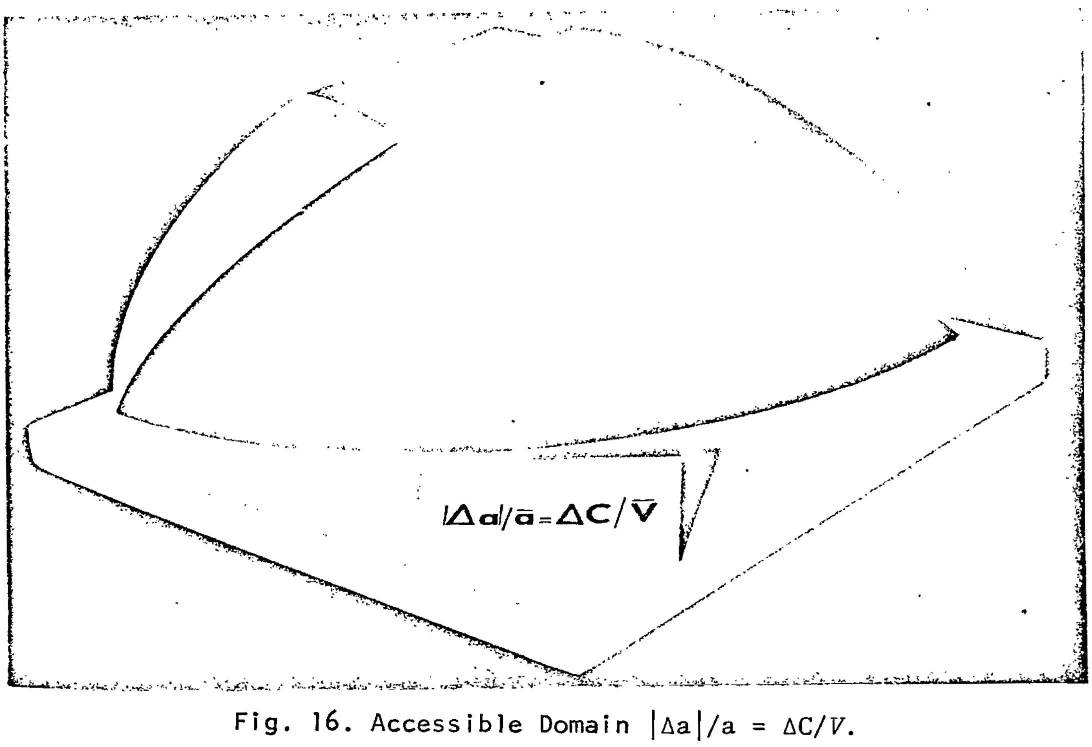

Fig. 16. Accessible Domain $|\Delta a|/a = \Delta C/V$.

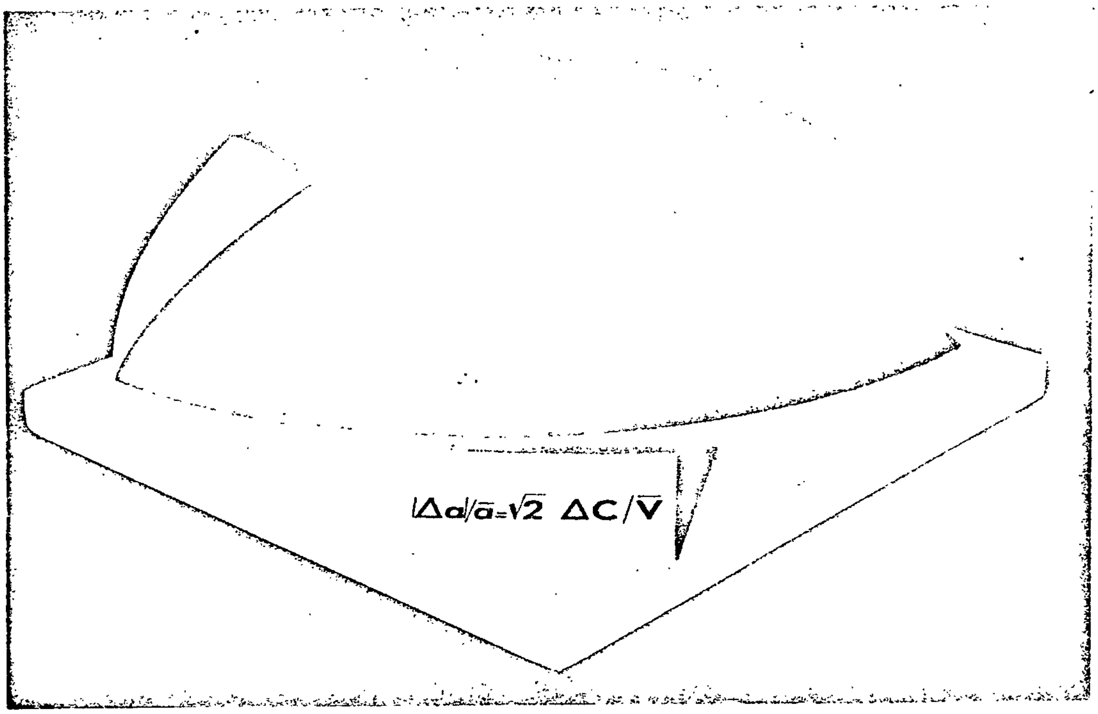

Fig. 17. Accessible Domain $|\Delta a|/a = \sqrt{2}\,\Delta C/V$.

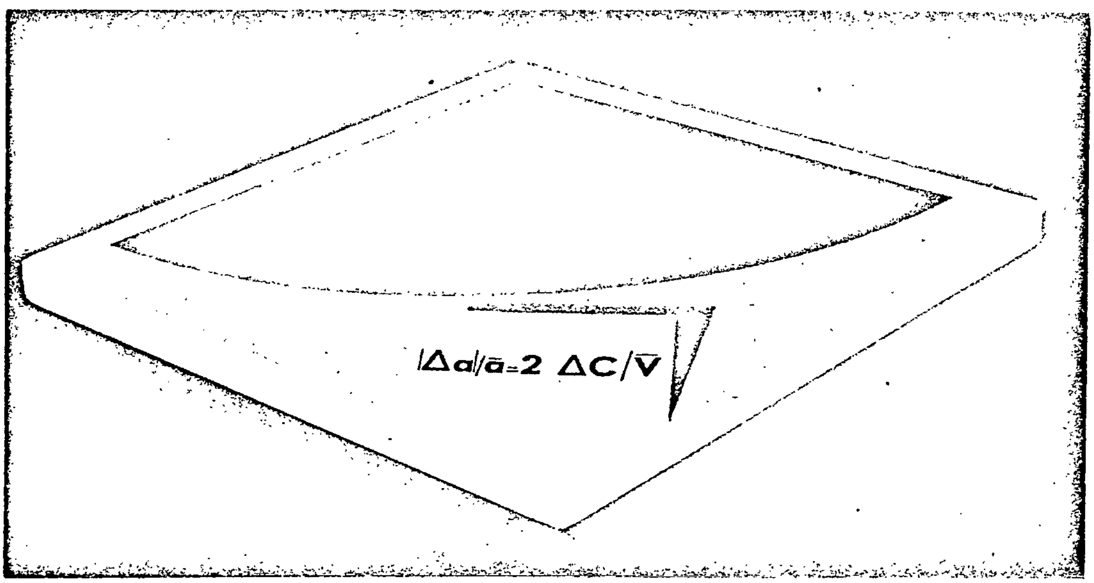

Fig. 18. Accessible Domain $|\Delta a|/a = 2\,\Delta C/V$.

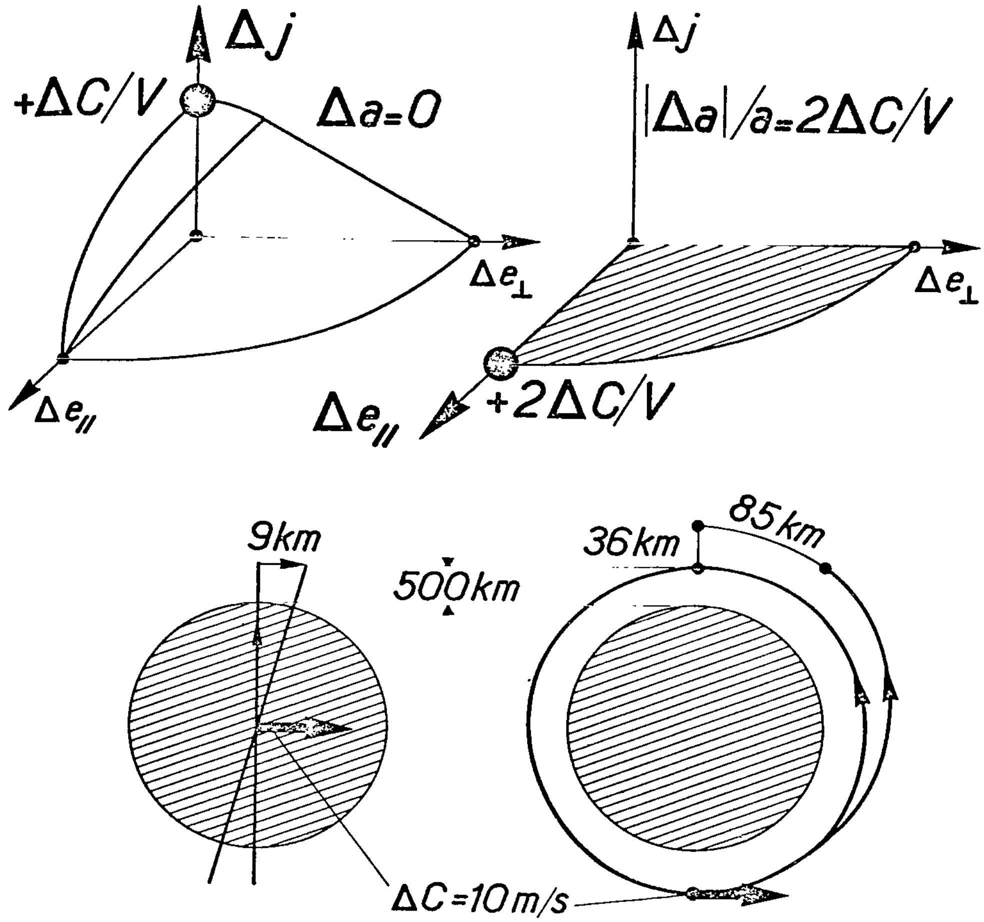

Fig. 19. Examples.

II,5.3.1. Solutions Close to the Solutions of Type I bis.

As in Chapter II,4., instead of the state component a, we shall use the state component:

$$B=\sqrt{\frac{\mu}{b}} \quad (\text{ here } \mu=1) \tag{25}$$

where b is the semi-minor axis.

The thrust acceleration $\vec{\gamma}$ applied at point M in the direction $\vec{D}$ produces the variation:

$$dB=-\frac{1}{2}\left(1-e^{2}\right)^{-3/4}\left\{e \sin v \vec{X}+\left[2+e(\cos v-\cos u)\right]\vec{Y}\right\}.\vec{D}\gamma\, dt=\vec{K}.\vec{D}\gamma\, dt. \tag{26}$$

In the expression of the vector of efficiency $\vec{\vec{p}}_V$, the term $2p_a\vec{V}$ must be replaced by $p_B\vec{K}$. The expressions of $\vec{r}$ and $\vec{V}$ are more complicated because of the fact that now e is $\neq 0$.

The solutions close to the solutions of type I bis in the case $\varepsilon \ll e \ll \ll 1$ are obtained for a kinematic adjoint $\vec{P}$ close to that relating to case I bis studied for $e = 0$.

Therefore let us posit:

$$P = \left[p_\xi\ ,\ p_\eta\ ,\ p_B = \varepsilon_B + \delta p_B\ ,\ p_\alpha\ ,\ p_\beta\right] \tag{27}$$

where $\varepsilon_B = \pm 1$ and where $|\delta p_B|, |p_\alpha|, |p_\beta|, |p_\xi|, |p_\eta|$ are $\ll 1$.

We shall posit:

$$M = \max\left(|\delta p_B|, |p_\alpha|, |p_\beta|, |p_\xi|, |p_\eta|, e\right). \tag{28}$$

The "vector of efficiency" $\vec{\vec{p}}_V$ has the components:

$$\vec{p_V}\begin{cases} X = p_\alpha \sin L - p_\beta \cos L - \frac{\varepsilon_B}{2}\, e \sin L + \text{order } M^2 \\ Y = -\varepsilon_B - \delta p_B + 2p_\alpha \cos L + 2p_\beta \sin L - \frac{\varepsilon_B}{4} e^2\left(1 + 2\cos^2 L\right) \\ \qquad + p_\alpha e \sin^2 L - p_\beta\, e \sin L \cos L + \text{order } M^3 \\ Z = -p_\eta \cos L + p_\xi \sin L + \text{order } M^2 \end{cases} \tag{29}$$

The direction of optimal thrust $\vec{D} = \vec{\vec{MP}}$ therefore is not far from axis $\vec{\vec{MY}}$. /151
On the other hand:

$$\begin{aligned} \vec{p_V}^2 &= 1 - 2\varepsilon_B\left(-\delta p_B + 2p_\alpha \cos L + 2p_\beta \sin L\right) + \text{order } M^2 \\ &\ 1 - 2\varepsilon_B\left[-\delta p_B + 2p_e \cos(L - L_e)\right] + \text{order } M^2 \end{aligned} \tag{30}$$

and positing as usual:

$$\begin{cases} p_\alpha = p_e \cos L_e \\ p_\beta = p_e \sin L_e \end{cases} \qquad \left(p_e = |\vec{p_e}| \geq 0\right) \tag{31}$$

a) If $p_e >> M^2$, $\vec{p}_V^{\,2}$ only admits the value 1 as a maximum for $L = L_e$ + order M and on condition that $\delta p_B = 2p_e$ + order M^2. There is only a single impulse per revolution, which restricts the solution to the case of intersecting orbits. Therefore the solution is not general.

b) Therefore we shall suppose the $|\delta p_B|$ and p_e are of the order M^2. Then:

$$\vec{p}_V^{\,2} = 1 + \frac{3e^2}{4}\left(1 + \cos^2 L\right) - 2\varepsilon_B\left[-\delta p_B + 2p_B\cos(L - L_e)\right] + p_z^2 \sin^2(L - L_z) + \text{order } M^3 \tag{32}$$

by positing:

$$\begin{cases} p_\xi = p_z \cos L_z \\ p_\eta = p_z \sin L_z \end{cases} \qquad (p_z = |\vec{p}_z| \geq 0) \tag{33}$$

Equation (32) is written again:

$$\vec{p}_V^{\,2} = 1 + \lambda + \mu\cos(L - L_e) + \nu\cos 2(L - w) + \text{order } M^3 \tag{34}$$

with

$$\lambda = \frac{9e^2}{8} + 2\varepsilon_B\,\delta p_B + \frac{p_z^2}{2} \tag{35}$$

$$\mu = -4\varepsilon_B p_e \tag{36}$$

$$\begin{cases} \nu\cos 2w = \frac{3}{8}e^2 - \frac{p_z^2}{2}\cos 2L_z \\ \nu\sin 2w = -\frac{p_z^2}{2}\sin 2L_z \qquad (\nu > 0) \end{cases} \tag{37}$$

$\vec{p}_V^{\,2}$ only admits the value 1 as a maximum for two values L' and L'' of L if this quantity is of the form:

$$\vec{p}_V^{\,2} = 1 - k\left[1 - \cos(L - L')\right]\left[1 - \cos(L - L'')\right] + \text{order } M^3 \tag{38}$$

an expression which depends on the three parameters k > 0, L', L''. Whence the two conditions of compatibility among the five parameters λ, μ, ν, L_e, w, occurring in (34):

$$w - L_e = (2p + 1)\frac{\pi}{2} + \text{order } M \tag{39}$$

$$\lambda = -\left(\frac{\mu^2}{8\nu} + \nu\right) + \text{order } M^3 \tag{40}$$

and the inequality: /152

$$\nu > \frac{\mu}{4} + \text{order } M^3 \tag{41}$$

expressing $k > 0$.

The application points of the impulses are therefore given by:

$$\left.\begin{array}{l} L' \\ L'' \end{array}\right\} = L_e \pm \operatorname{Arc} \cos \frac{\mu}{4\nu} + \text{order } M. \tag{42}$$

Their straight ascents are near-symmetrical in relationship to L_e.

The bi-impulsional solutions thus obtained are of a general type, contrary to the mono-impulsional solutions previously found, since seven parameters are available (p_ξ, p_η, δp_B, p_α, p_β and the magnitudes $I'\Delta V$ and $I''\Delta C$ of the two impulses) reduced to five independent parameters by the two equations (39) and (40), in order to realize the five variations of the orbital elements.

The optimal directions of the impulses are given by:

$$\vec{P_V} \begin{cases} X = -\frac{\varepsilon_B}{2} e \sin L + \text{order } M^2 \\ Y = -\varepsilon_B - \delta p_B + 2p_e \cos(L - L_e) - \frac{\varepsilon_B e^2}{4}(1 + 2\cos^2 L) + \text{order } M^3 \\ Z = p_z \sin(L - L_z) + \text{order } M^2 \end{cases} \tag{43}$$

1/ If $p_z >> e$ (therefore $M = p_z$), we deduce from (35) and (37):

$$\nu \simeq \frac{p_z^2}{2} > 0 \tag{44}$$

$$w \simeq L_z \pm \frac{\pi}{2} \tag{45}$$

$$\lambda \simeq 2\varepsilon_B \delta p_B + \frac{p_z^2}{2}. \tag{46}$$

The conditions of compatibility (39) - (41) are then written:

$$p_z^2 > -2\varepsilon_B p_e \tag{47}$$

$$L_e \simeq L_z + q\pi \quad (q \text{ integer}). \tag{48}$$

Therefore the vectors $\vec{\vec{p}}_e$ and $\vec{\vec{p}}_z$ are colinear.

$$\varepsilon_B \, \delta p_B \simeq -\left(2 \frac{p_e^2}{p_z^2} + p_z^2\right) < 0. \tag{49}$$

The two impulses are applied at points:

$$\left.\begin{matrix} L' \\ L'' \end{matrix}\right\} \simeq L_e \pm \operatorname{Arc\,cos}\left(-2\,\varepsilon_B \frac{p_e}{p_z^2}\right), \tag{50}$$

symmetrical in relationship to $\vec{\vec{p}}_e$ and $\vec{\vec{p}}_z$.

In particular, for $p_e \ll p_z^2$, the two impulses are diametrically opposed: /153

$$\left.\begin{matrix} L' \\ L'' \end{matrix}\right\} \simeq L_e + \frac{\pi}{2} \tag{51}$$

and

$$\varepsilon_B \, \delta p_B \simeq -p_z^2 < 0. \tag{52}$$

The bi-impulsional solutions thus obtained for $p_z \gg e$ could be called *type I, in analogy with the study of* § II,5.2.1.3.

2/ If, on the contrary, $p_z \ll e$ (therefore M = e), we obtain by (35) and (37):

$$w \simeq k\frac{\pi}{2} \tag{53}$$

$$\nu \simeq (-1)^k \frac{3}{8} e^2 > 0 \quad \text{therefore} \quad k = 2q \text{ et } w \simeq q\pi \tag{54}$$

$$\lambda \simeq \frac{9}{8} e^2 + 2\varepsilon_B \delta p_B. \tag{55}$$

Thus the conditions of compatability (39) - (41) are written:

$$\frac{3}{8} e^2 > -\varepsilon_B p_e \tag{56}$$

$$L_e \simeq (2r+1)\frac{\pi}{2} \tag{57}$$

$$\varepsilon_B \, \delta p_B \simeq -\left(\frac{8}{3} \cdot \frac{p_e^2}{e^2} + \frac{3e^2}{4}\right) < 0. \tag{58}$$

The two impulses are applied at points:

$$\left.\begin{matrix} L' \\ L'' \end{matrix}\right\} \simeq (2r+1)\frac{\pi}{2} \pm \mathrm{Arc}\cos\frac{8\varepsilon_B p_e}{3e^2}. \tag{59}$$

symmetrical in relation to the minor axis.*

In particular, for $p_e \ll e^2$, the two impulses are diametrically opposed and applied at the perigee and apogee of the orbit, and:

$$\varepsilon_B\,\delta p_B \simeq -\frac{3}{4}e^2 < 0. \tag{60}$$

The solutions thus obtained for $p_z \ll e$ could be called *type I bis, by analogy with the study of* § II,5.2.1.3., *although here there is no more degeneracy of the linearized solution.*

c) Equation (34) shows that solutions with *three impulses* or more can only be found if λ, μ and ν are simultaneously of order M^3, while still heeding the definition $M = \max(p_z, e)$, This is not possible unless: /154

$$\left.\begin{matrix} L_z = \begin{cases} 0 \\ \pi \end{cases} + \text{order}\,M & (61) \\ p_z = \frac{\sqrt{3}}{2}e + \text{order}\,M^2 & (62) \end{matrix}\right\} \text{which assures that } \nu = \text{order } M^3$$

$$\varepsilon_B\,\delta p_B = -\frac{3}{4}e^2 + \text{order}\,M^3 \tag{63}$$

$$p_e \doteq \mathit{ordre}\ M^3 \tag{64}$$

Then the direction of impulse is such that:

$$\frac{Z}{X} = \mp\,\varepsilon_B\sqrt{3} + \text{order}\,M. \tag{65}$$

Therefore this direction is approximately contained in one of the planes (π^+) or (π^-) going through $\overrightarrow{\overrightarrow{MY}}$ and forming angles of ±30° with the local horizontal plane.

---

* Note that for $p_z = 0$ (coplanar case) we find again the commutation ψ_s [13]. The commutation ψ_i is found in the plane type II.

These solutions, of a new type, will not be studied in detail.

II,5.3.2. Solutions Close to the Solutions of Type III.

These solutions are found in the case $\varepsilon << e << 1$ for a kinematic adjoint $\vec{P}$ close to that referring to case III studied for e = 0.

$$\vec{P} \begin{cases} p_\xi = p_z \cos L_z = \left(\frac{\sqrt{3}}{2} + \delta p_z\right) \cos L_z \\ p_\eta = p_z \sin L_z = \left(\frac{\sqrt{3}}{2} + \delta p_z\right) \sin L_z \\ p_a \\ p_\alpha = p_e \cos L_e = \left(\frac{1}{2} + \delta p_e\right) \cos L_e \\ p_\beta = p_e \sin L_e = \left(\frac{1}{2} + \delta p_e\right) \sin L_e \end{cases} \tag{66}$$

Since the nominal orbit (O) is no longer circular, it is no longer possible to a priori impose $p_\beta = 0$, that is to say $L_e = 0$ with the choice of axes indicated in Figure I,3. -1.

We shall posit:

$$L_z = L_e + \begin{cases} 0 \\ \pi \end{cases} + \delta L \quad \text{according to} \quad \Delta e_\perp) = \begin{cases} - \\ + \end{cases} \tag{67}$$

where $\delta L = 0$ for solutions of type III of case e = 0.

We shall also posit:

$$M = \max\left(e, |p_a|, |\delta p_e|, |\delta p_z|, |\delta L|\right) \tag{68}$$

Then the "vector of efficiency" has as components: /155

$$\vec{P_V}\begin{cases} X = \frac{1}{2}\sin(L - L_e) + \delta p_e \sin(L - L_e) + \text{order } M^2 \\ Y = \cos(L - L_e) + \frac{e}{2}\sin L \sin(L - L_e) + 2\delta p_e \cos(L - L_e) + \text{order } M^2 \\ Z = \pm\left[\frac{\sqrt{3}}{2}\sin(L - L_e) - \frac{\sqrt{3}}{2} e \cos L \sin(L - L_e) + \delta p_z \sin(L - L_e) \right. \\ \qquad\qquad \left. - \frac{\sqrt{3}}{2}\delta L \cos(L - L_e) + \text{order } M^2\right] \end{cases} \tag{69}$$

The direction of optimal thrust is situated (at approximately order M) in one of the planes (π^+) or (π^-) going through $\overrightarrow{\overrightarrow{MY}}$ and forming angles of ±30° with the local horizontal plane X = 0 ($\pm = -\Delta e_\perp$).

This direction coincides with $\overrightarrow{\overrightarrow{MY}}$ (at approximately order M) for $L = L_e$ and $L = L_e + \pi$.

On the other hand:

$$\vec{P_V}^2 = 1 + \lambda + \mu \cos(w - w_1) + \nu \cos 2(w - w_2) + \frac{e}{8}\cos(3w + L_e) + \text{order } M^2 \tag{70}$$

by positing:

$$\boxed{w = L - L_e} \tag{71}$$

$$\lambda = \frac{5}{2}\delta p_e + \frac{\sqrt{3}}{2}\delta p_z \tag{72}$$

$$\begin{cases} \mu \cos w_1 = 4 p_a - \frac{e}{8}\cos L_e \\ \mu \sin w_1 = \frac{11}{8} e \sin L_e \end{cases} \quad (\mu > 0) \tag{73}$$

$$\begin{cases} \nu \cos 2 w_2 = \frac{3}{2}\delta p_e - \frac{\sqrt{3}}{2}\delta p_z \\ \nu \sin 2 w_2 = -\frac{3}{4}\delta L \end{cases} \quad (\nu > 0) \tag{74}$$

The datum of the five parameters p_a, δp_e, L_e, δp_z, δL is equivalent to the datum of the five parameters λ, μ, ν, w_1, w_2.

a) If e, $|\delta L|$ and $|\sqrt{3}\ \delta p_e - \delta p_z|$ are << M (and therefore ν << M), for example if these parameters are of the order M^2, $\vec{\vec{p}}_v^{\,2}$ can only have the value 1 as a maximum for a single value of w:

$$w = w_1 + \text{order } M = \text{order } M. \tag{75}$$

on condition that: /156

$$\lambda + \mu = \frac{5}{2}\,\delta p_e + \frac{\sqrt{3}}{2}\,\delta p_z + 4\,|p_a| + \text{order } M^2 = \text{order } M^2. \tag{76}$$

These mono-impulsional solutions are not of a general type (intersecting orbits). They could be called solutions of type (3) ("triple point" N) by analogy with the study of § II,5.2.1.3. (Figure 11b).

b) If e << M (for example e = order M^2) and if, moreover, the conditions:

$$\nu > \frac{\mu}{4} + \text{order } M^2 \quad c.a.d. \quad \nu > |p_a| + \text{order } M^2 \tag{77}$$

$$w_2 - w_1 = (2p+1)\frac{\pi}{2} + \text{order } M \quad c.a.d \quad w_2 = (2p+1)\frac{\pi}{2} + \text{order } M \tag{78}$$

$$\lambda = -\left(\frac{\mu^2}{8\nu} + \nu\right) + \text{order } M^2 \tag{79}$$

are satisfied, $\vec{\vec{p}}_v^{\,2}$ allows the value 1 as a maximum for two given values of w by:

$$\left.\begin{matrix} w' \\ w'' \end{matrix}\right\} = w_1 \pm \text{Arc cos}\,\frac{\mu}{4\nu} + \text{order } M = \pm\,\text{Arc cos}\,\frac{\mu}{4\nu} + \text{order } M \tag{80}$$

as reasoning analogous to that in § II,5.3.1. shows.

The bi-impulsional solutions obtained in this way are of a general type,in contrast to the mono-impulsional solutions. In fact there are available seven parameters (p_a, δp_e, L_e, δp_z, δL and the magnitudes I' ΔC and I'' ΔC of the two impulses) reduced to five independent parameters by the two conditions of compatibility (78) (79), in order to realizé the five variations of the orbit elements.

The two impulses are symmetrical in relation to the direction of any argument L_e. The solutions obtained in this way could be called type (4) (transition III $\leftrightarrow$ I) by analogy with the study of § II,5.2.1.3. (Figure 11b).

Let us note that (78) entails:

$$\delta L = \text{order } M^2 \tag{81}$$

and therefore

$$\nu = \left|\frac{3}{2}\,\delta p_e - \frac{\sqrt{3}}{2}\,\delta p_z\right| \tag{82}$$

so that equation (79) is also written:

$$\left(\delta p_e + \sqrt{3}\cdot\delta p_z\right)\left(\sqrt{3}\,\delta p_e - \delta p_z\right) = \frac{4\,p_a^2}{\sqrt{3}}. \tag{83}$$

If $|p_a| \ll M$ (for example $|p_a|$ = order M^2), then $\left.\begin{matrix} w' \\ w'' \end{matrix}\right\} = \pm\frac{\pi}{2}$ + order M.

The two impulses are almost diametrically opposed. These solutions could be called type (13) (transition III $\leftrightarrow$ II) by analogy with the study of § II,5.2.1.3. (Figure 11b).

c) If e = order M, the presence of the term in $\cos(3w + L_e)$ in (70) shows that there can exist cases where $\vec{\vec{p}}_V^{\,2}$ allows the value 1 as a maximum for three values of w, i.e. *tri-impulsional* solutions may exist.

For this it is necessary for $\vec{\vec{p}}_V^{\,2}$ to be in the form:

$$\vec{p}_V^{\,2} = 1 - k\left[1 - \cos(w - w')\right]\left[1 - \cos(w - w'')\right]\left[1 - \cos(w - w''')\right] \tag{84}$$

where w', w'', w''' are relative to the three impulses and where $k > 0$. /157

The identification of the coefficients of the terms in cos $3w$, sin $3w$ and sin w in (70) and (84) leads to:

$$k = \frac{e}{2} > 0 \tag{85}$$

$$\boxed{\begin{aligned} & w' + w'' + w''' + L_e = 0 && (86) \\ & \sin w' + \sin w'' + \sin w''' + \sin w' \cdot \sin w'' \cdot \sin w''' = 3 \sin L_e && (87) \end{aligned}}$$

Let us note that it is still possible to write the vectorial equation (19) and to use Figure 5 while committing a relative error of the order of e. Therefore the angle L_e, with the exception of e, is the straight ascent of the "consumption vector" $\vec{\vec{\Delta C}}$ defined by (19). Therefore L_e is well determined for a given transfer.

Figure 20 concerns the resolution of the system (86) (87) referred to w', w'', w''' for different values of L_e.

The abscissae, counted on the axes indicated, at the foot of the perpendiculars dropped from M onto these axes are respectively equal to /158

$$w' + \frac{L_e}{3},\ w'' + \frac{L_e}{3},\ w''' + \frac{L_e}{3}.$$

The locus of point M when w', w'', w''', solutions of the system (86) (87) vary with L_e remaining definite, is a line in the plane thus defined. It is possible to limit oneself to the case $0 \leq L_e \leq 90°$ and to the interior of the equilateral triangle OAB. Then it is necessary to complete by symmetry referring to the sides of the triangle and the sides of the successive triangles obtained in this way.

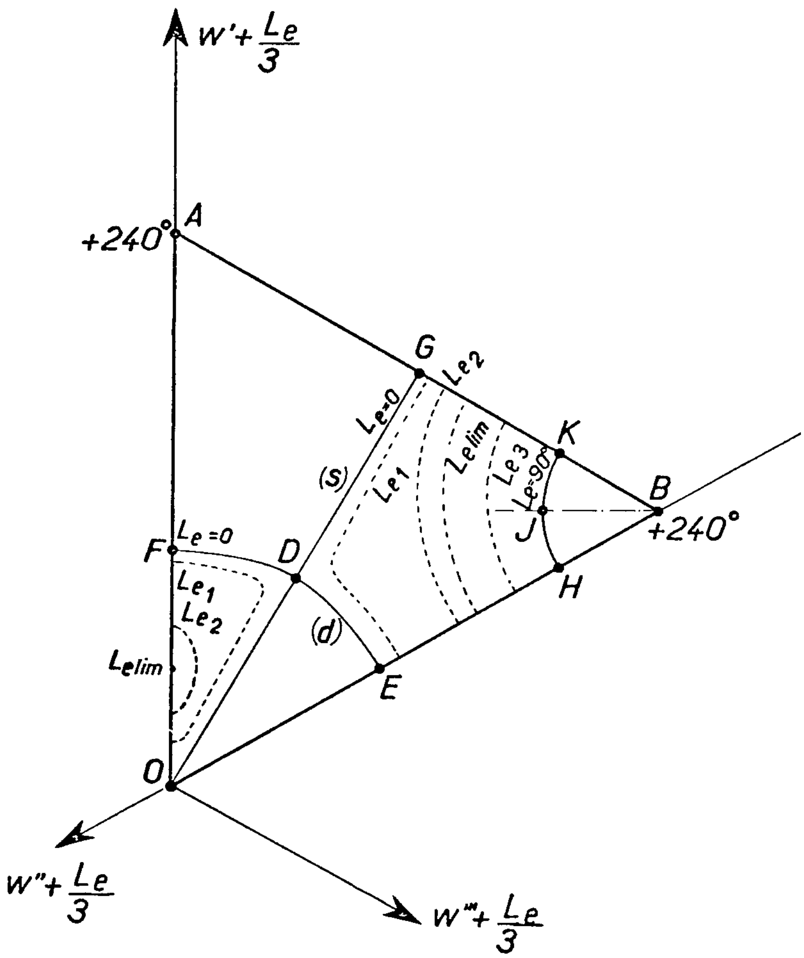

Fig. 20.

Only the lines $L_e = 0$ and $L_e = 90°$ in solid lines have been calculated. The plot of the other lines, dotted, is only probable.

a) Case $L_e = 0$.

The "consumption vector" $\vec{\vec{\Delta C}}$ is parallel to the major axis of *(O)*.

There are two types of tri-impulsional solutions:

1. SYMMETRICAL SOLUTIONS (segment OG of Figure 20).

$$L''' = 0 \qquad (88)$$

$$L'' = -L'. \tag{89}$$

These solutions consist of applying an impulse tangent to the perigee P ($L''' = 0$) of the orbit (O) and two impulses to point $M'(L')$ and $M''(L'')$, symmetrical in relation to the major axis, practically contained in one of the planes (π^+) or (π^-) (Figure 21b).

According to the relative magnitudes of these impulses it is possible to attain any point G inside triangle PV'V'' (Figure 21a). G is the "center of gravity" of the ficticious "masses" placed in V', V'' and P in proportion to the corresponding impulses.

The third order cone of peak P, based on Viviani's Window (V) and equation:

$$y_1(x_1^2+y_1^2) - 2(\Delta C - z_1)(x_1^2+y_1^2) + y_1(\Delta C - z_1)^2 = 0 \tag{90}$$

limits to the inside of the total volume (V) a volume corresponding to the symmetrical tri-impulsional solutions.

In order to visualize this volume better, let us consider its sections $y_1 = C^{te}$ produced by the segment $v'v''$. The limitant $\widehat{V_1V_3}$ $\widehat{V_2V_4}$ of equation:

$$x_1^2 = y_1 \frac{(\Delta C - y_1 - z_1)^2}{2\Delta C - y_1 - 2z_1} \qquad (y_1 = \text{constant}) \tag{91}$$

is symmetrical in reference to $x_1 = 0$, and admits of a double point in $z_1 = \Delta C - y_1$, an asymptote $z_1 = \Delta C - \frac{y_1}{2}$ and two parabolic branches parallel to the axis $\vec{Oz}_1$.

2. DI-SYMMETRICAL SOLUTIONS.

These correspond to the near-circular arc $\widehat{EF}$ of Figure 20, symmetrical in relation to OG.

$$L' + L''' = -L'' \tag{92}$$

$$L' - L''' = 2\,\mathrm{Arc}\cos\left(\sin\frac{L''}{2}\,\mathrm{tg}\,\frac{L''}{2}\right). \tag{93}$$

When L'' varies from -90° to -103°40', (arc $\widehat{DE}$ of Figure 20) the point V'' describes the arc $\widehat{LS}$ of (V) (Figure 22a), while points V' and V'' respectively describe the arcs $\widehat{LT}$ and $\widehat{PT}$. Point T corresponds to angle L = 51°50'.

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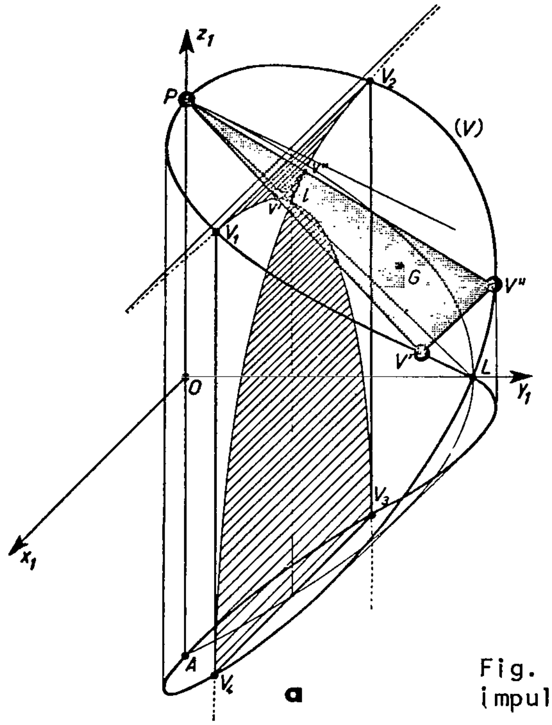


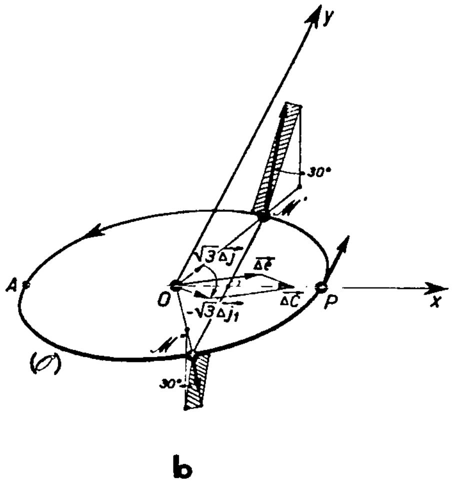
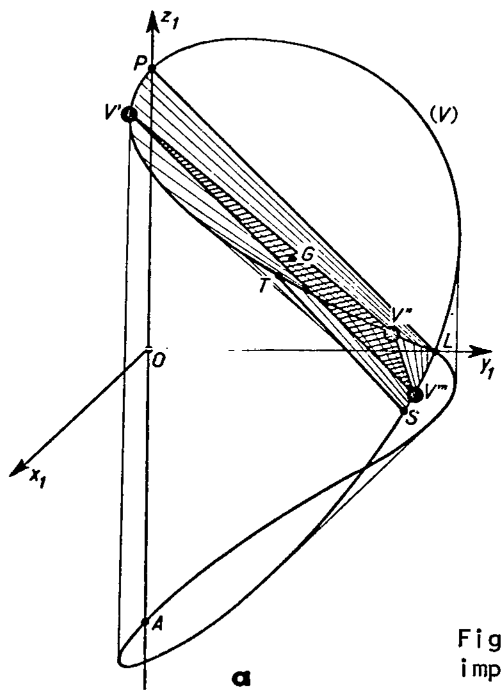


Fig. 21. Type III.Symmetrical Tri-impulsional Solutions·(L_e = 0, $\Delta e_{\perp}$ < 0).



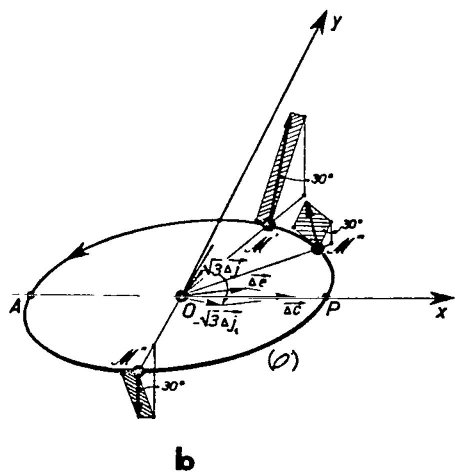


Fig. 22. Type III. Di-symmetrical Tri-impulsional Solutions. (L_e =, $\Delta e_{\perp}$ < 0).

By a suitable choice of the impulse magnitudes in *M'*, *M''*, *M'''*, (Figure 22b), it is posible to reach any point G inside triangle V'V''V'''. /160

When L'' varies, this triangle entails a volume limited by its ruled surface produced by segments V''V''', V'''V', V'V''. This volume and the volume symmetrical in relationship to plane $x_1 = 0$ correspond to the di-symmetrical tri-impulsional solutions.

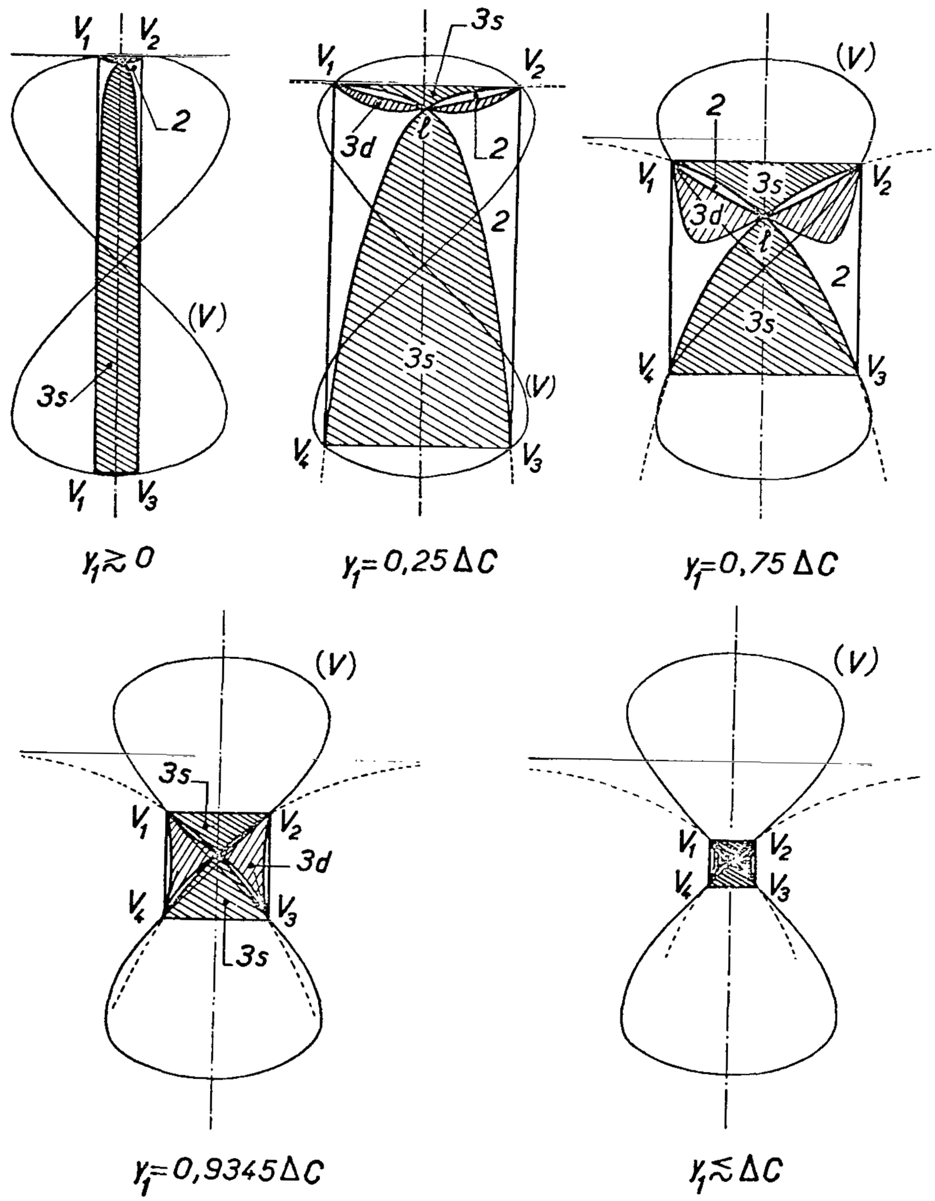


Fig. 23. Type III. Sections $y_1 = c^{te}$.

Figure 23 summarizes the previous results by showing, for each section $y_1 = C^{te}$, the zones corresponding to the optimal two impulse solutions (2) or to three symmetrical (3S) or di-symmetrical (3d) impulses.

The frontier of zone (3d) admits of angular points in V_1, V_2 and l. For /161
$y_1 = y_{1s} = 0.9345\ \Delta C$, points V_3 and V_4 are turning points. For $0.9345\ \Delta C < y_1 \leq \Delta C$, they are angular points.

It can be noted that for $y_1 \lessapprox \Delta C$, zones (2) with two impulses are very reduced.

This case corresponds to a rotation $\vec{\vec{\Delta j}}$ around an axis close to "parameter" $\vec{\vec{Oy}}$, associated with a shift of the center in a direction close to the major axis and with a weak "dilatation". Then the three impulse solutions present a small impulse tangent to the perigee (3S) or almost tangent in the vicinity of the perigee (3d) and two other impulses, of which at least one is significant, almost perpendicular to the tangent at the orbit, applied at points close to the peaks of the "parameter", and symmetrical in relation to the major axis (3S) or almost diametrically opposed (3d).

b) Case $L_e = 90°$

The "consumption vector" $\vec{\vec{\Delta C}}$ is borne by the "parameter" $\vec{\vec{Oy}}$ of *(O)*.

The three impulses are situated near the apogee (Figure 24b).

When the near-circular arcs $\overset{\frown}{HJ}$, $\overset{\frown}{JK}$ of Figure 20 are described, symmetrical in relationship to BJ, points V'V''V''' respectively describe the arcs $\overset{\frown}{FA}$ $\overset{\frown}{AD}$, $\overset{\frown}{EQ}$ $\overset{\frown}{QD}$ and $\overset{\frown}{FS}$ $\overset{\frown}{SC}$ (Figure 24 a) with:

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C	$W = 40°34'$	$L = 130°34'$
S	$W = 47°03'$	$L = 137°03'$
F	$W = 65°12'$	$L = 155°12'$
A	$W = 90°$	$L = 180°$
D	$W = 114°48'$	$L = 204°48'$
Q	$W = 132°57'$	$L = 222°57'$
E	$W = 139°36'$	$L = 229°36'$

Then triangle V'V''V''' produces a volume limited by the ruled surface produced by the segments V''V''', V'''V' and V'V''.

This volume corresponds to the tri-impulsional solutions of case $L_e = 90°$. It may be noticed that this volume is not as significant as the one which refers to case $L_e = 0°$.

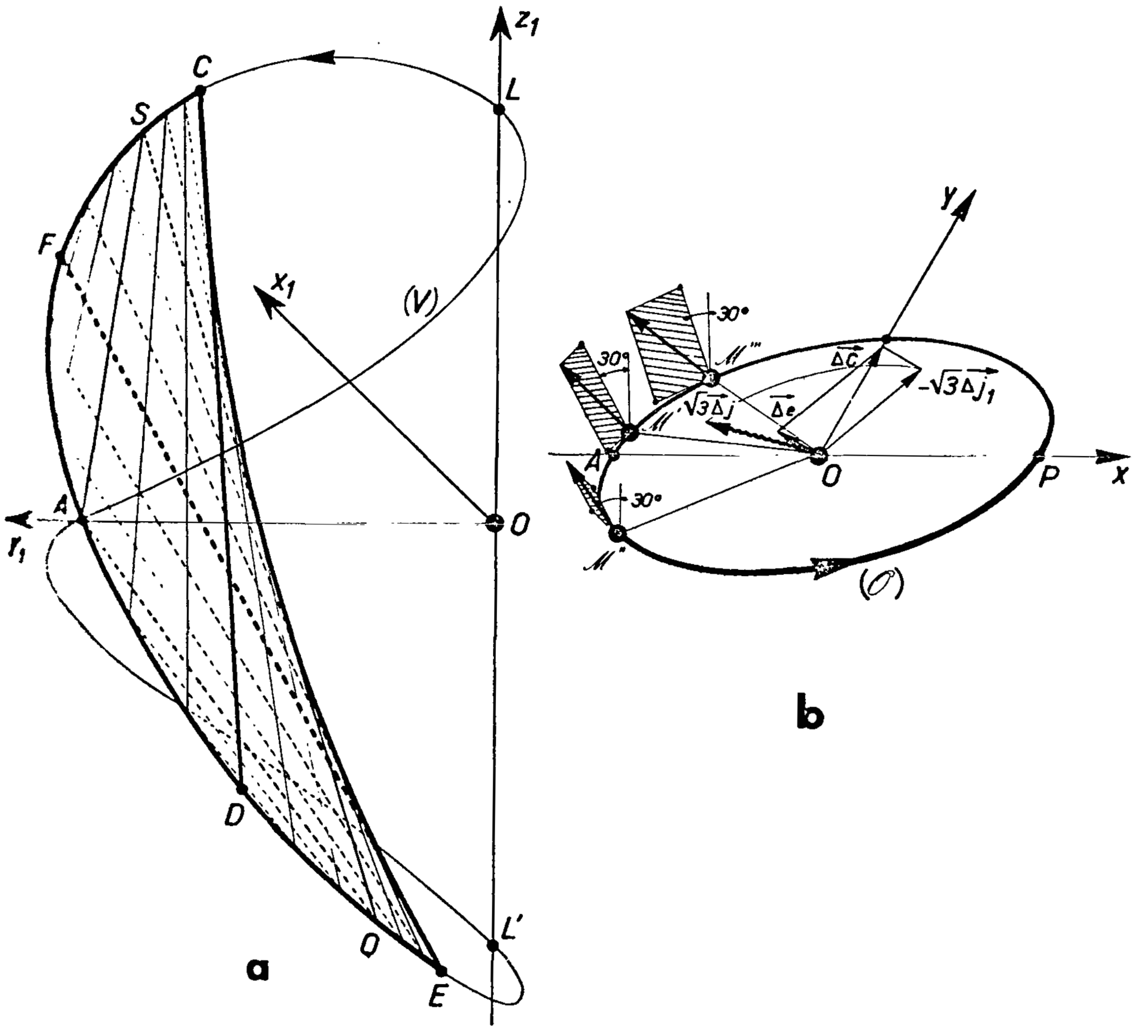

Fig. 24. Type III. Tri-impulsional Solutions. ($L_e = 90°$, $\Delta e \perp < 0$).

II,5.4. CONCLUSION.

The complete analytical solution of the first order, referring to the problem of economic transfers (minimal characteristic velocity, indifferent duration) between close, near-circular orbits, co-planar or not, was obtainable by linearization around a nominal circular orbit.

Such transfers can always be realized in an optimal way with the help of two well-determined impulses. Nevertheless, in certain cases (degeneracy of the linearized solution) a more extended choice is offered, and the same transfer may be able to be achieved in various ways, since the corresponding consumptions differ only by quantities of the second order referring to variations in the orbit elements.

However, this degeneracy disappears if the transfer takes place around orbits with an eccentricity smaller than one,but larger than the size of the transfer, and then the solutions are mono-, bi- or tri-impulsional.

II,6. OPTIMAL IMPULSE LONG DURATION RENDEZVOUS BETWEEN CLOSE NEAR-CIRCULAR ORBITS, COPLANAR OR NON-COPLANAR.

II,6.1. INTRODUCTION.

In addition to the five imposed variations $\Delta\xi$, $\Delta\eta$, Δa, $\Delta\alpha$, $\Delta\beta$ of the orbital elements considered in Chapter II,5. in the case of transfer, the sixth variation $\Delta\tau$ referring to the rendezvous must be introduced here.

Let us recall that, for any e, $\Delta\tau$ = (temporal lag δt_o, counted on the final orbit (O_f), of the fictitious mobile $\overline{M}_o$ of the same straight ascent L_o as the initial real mobile M_o, referring to the initial target mobile M_{co}) + $\frac{3}{2} M_o \, \Delta a + \overset{\rightrightarrows}{R}_o \cdot \overset{\rightrightarrows}{\Delta e}$ (Figure 1). M_o and $\overset{\rightrightarrows}{R}_o$ can depend only on e and L_o. Here, e = 0, and the *temporal* lag δt_o can be replaced by the *angular* lag $\widehat{M_{co} O M_o}$. Furthermore $M_o = L_o$ and $\overset{\rightrightarrows}{R}_o = \overset{\rightrightarrows}{2y}_o$. /163

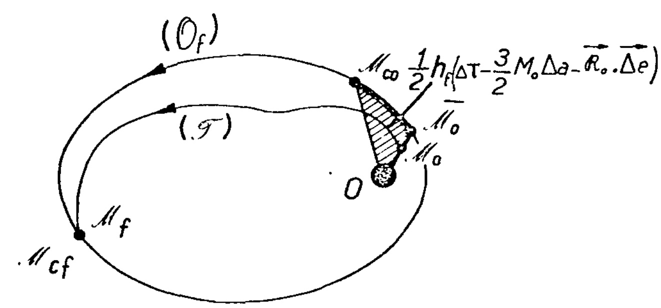

Fig. 1. Interpretation of the Parameter $\Delta\tau$.

The linearized equations which permit these six variations to be calculated are written, in the order approximately relative to ε (see Appendix 3):

$$\left\{\begin{aligned} \xi' &= \gamma_z \sin L \\ \eta' &= -\gamma_z \cos L \\ a' &= 2\gamma_y \\ \alpha' &= \gamma_x \sin L + 2\gamma_y \cos L \\ \beta' &= -\gamma_x \cos L + 2\gamma_y \sin L \\ \tau' &= -2\gamma_x + 3L\gamma_y \end{aligned}\right. \tag{1}$$

The efficiency vector has as components in the turning axes:

$$\vec{P_V}\begin{cases} X = p_e \sin(L - L_e) - 2p_\tau \\ Y = \underbrace{2p_a + 3\, p_\tau\, L}_{2p_{a_1}} + 2p_e \cos(L - L_e) \\ Z = p_Z \sin(L - L_Z) \end{cases} \tag{2}$$

Continuing, we shall suppose that the rendezvous is of long duration, i.e. that the transfer angle ΔL is $\gg 2\pi$.

If we then admit a relative error in the solution, no longer of order ε but of order: max $(\varepsilon, \frac{1}{\Delta L})$, the equation in τ' may be written:

$$\tau' = 3L\gamma_Y \simeq 3(2k\pi)\gamma_Y \tag{3}$$

where k = (turn number) $\leq$ N = (total number of turns).

Then the efficiency vector becomes:

$$\vec{P_V}\begin{cases} X = p_e \sin(L - L_e) \\ Y = \underbrace{2p_a + 3p_\tau L}_{2p_{a_1}} + 2p_e \cos(L - L_e) \\ Z = p_Z \sin(L - L_Z) \end{cases} \tag{4}$$ /164

In this approximation it is important to note that the straight ascent L intervenes only by the turn number k, which simplifies the solutions a great deal.

Let us also note that since only the impulsional solutions (or strictly speaking, the singular solutions) are considered here, $|\vec{p}_V|$ is ≤ 1; therefore, in (2), $|y|$ is $\leq$ 1, which brings about: $p_\tau \leq$ order $\frac{1}{\Delta L}$.

II,6.2. ACCESSIBLE DOMAIN.

This is a matter of defining the totality of the points of the five dimensional space of the variations $\Delta e_{//}$, $\Delta e_\perp$, Δj, Δa, $\Delta\tau$ which can be reached with the characteristic velocity ΔC.

In the case of long-duration rendezvous, it is convenient to take, no longer $\Delta\tau$, as the sixth variation but:

$$\Delta \mathcal{T} = -\frac{4}{3} \frac{\Delta \tau}{\Delta L} + \Delta a = -\frac{4}{3} \frac{\delta t_0}{\Delta L} + \Delta a + \text{order } \frac{1}{\Delta L} \tag{5}$$

which introduces supplementary symmetries into the accessible domain.

As a matter of fact, observing that:

$$\mathcal{T}' = \frac{4}{3} \frac{\tau'}{\Delta L} + a' = \left(1 - 2\frac{k}{N}\right) a' \tag{6}$$

and taking the axis of reference $\vec{Ox}$ according to the line of the nodes, the support of the rotation vector $\vec{\Delta j}$ ($d\alpha$ and $d\beta$ then respectively represent $de_{//}$ and $d_{e\perp}$), it is possible to modify the sign of each of these five variations defining the rendezvous, *without modifying the other variations*, by making the following changes for every thrust:

$\Delta e_{//}, \Delta e_{\perp}, \Delta j, \Delta a, \Delta \mathcal{T}$	L	k	X	Y	Z
$-\Delta e_{//}$	$\pi - L$	k	$-X$	Y	$-Z$
$-\Delta e_{\perp}$	$-L$	k	$-X$	Y	Z
$-\Delta j$	$\pi + L$	k	X	Y	Z
$-\Delta a$	$\pi + L$	$N-k$	$-X$	$-Y$	$-Z$
$-\Delta \mathcal{T}$	L	$N-k$	X	Y	Z

Since these changes do not modify the magnitude of the thrusts, and thus consumption, the accessible domain is symmetrical in relation to the axes, planes and coordinate spaces, and it is enough to consider only the positive variations. (However it is sometimes convenient to consider any sign variations in order to show how the optimal solutions are linerally composed). /165

Let us first attempt to define, in the plane Δa, $\Delta \mathcal{T}$, the totality of points accessible with the characteristic velocity ΔC, *with the variations* $\Delta e_{//}$, $\Delta e_{\perp}$, Δj *being fixed.*

This presumes that the characteristic velocity ΔC is at least equal to the minimal characteristic velocity ΔC_{min} necessary to produce the variations $\Delta e_{//}$, $\Delta e_{\perp}$, Δj which define point G of Figure 2, or:

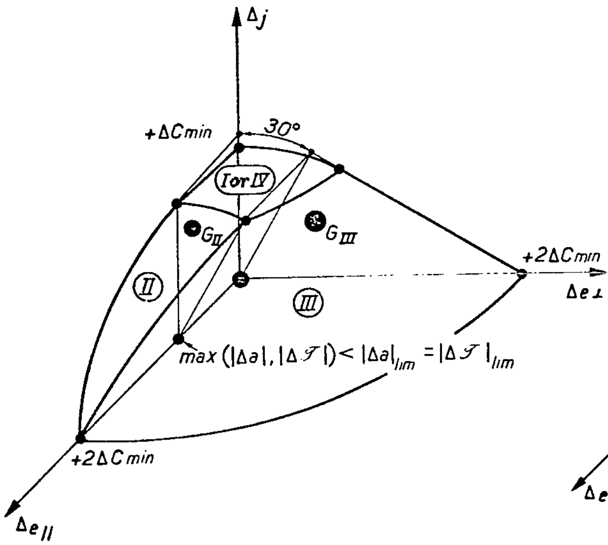

Fig. 2. Domain Accessible For:

$\max(|\Delta a|, |\Delta\mathcal{T}|) < |\Delta a|_{\mathrm{lim}} = |\Delta\mathcal{T}|_{\mathrm{lim}}$.

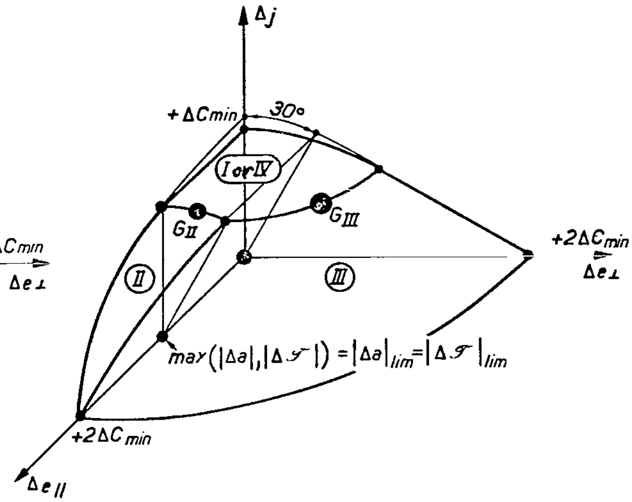

Fig. 3. Domain Accessible For:

$\max(|\Delta a|, |\Delta\mathcal{T}|) = |\Delta a|_{\mathrm{lim}} = |\Delta\mathcal{T}|_{\mathrm{lim}}$.

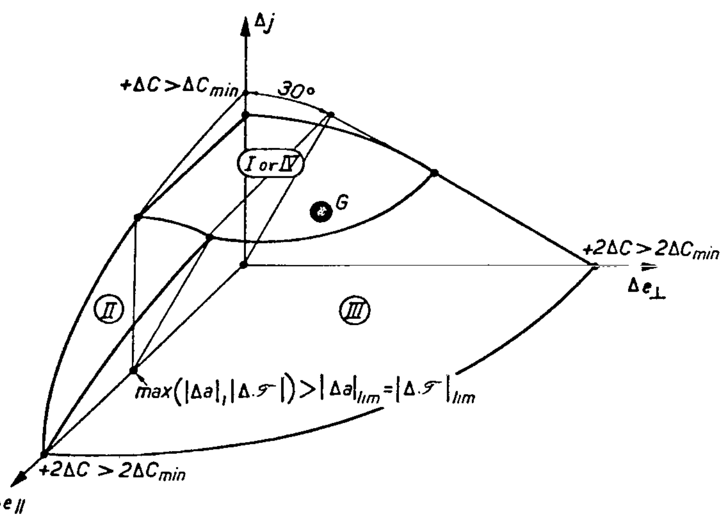

Fig. 4. Domain Accessible For:

$\max(|\Delta a|, |\Delta\mathcal{T}|) > |\Delta a|_{\mathrm{lim}} = |\Delta\mathcal{T}|_{\mathrm{lim}}$.

$$\Delta C^2 \geqslant \Delta C^2_{min} = \begin{cases} \dfrac{\Delta e_{\parallel}^2}{4} + \Delta e_{\perp}^2 + \Delta j^2 \text{ si } 3\Delta e_{\perp}^2 \leqslant \Delta j^2 \ (II) \\ \\ \dfrac{\Delta e_{\parallel}^2}{4} + \dfrac{(|\Delta e_{\perp}| + \sqrt{3}|\Delta j|)^2}{4} \text{ si } 3\Delta e_{\perp}^2 \geqslant \Delta j^2 \ (III) \end{cases} \quad (7)$$ /166

There then exists, associated with ΔC, a maximal value $|\Delta a|_{max}$ of $|\Delta a|$ (Figure 4) obtained in the study of simple transfer (ΔT indifferent). Δa^2_{max} is the largest of the roots of the biquadratic equation (II,5. - 22).

The domain accessible in the plane Δa, ΔT is therefore necessarily contained in the band: $-|\Delta a|_{max} \leq \Delta a \leq +|\Delta a|_{max}$ (Figure 5). The segment: $\Delta T = 0$, $-|\Delta a|_{max} \leq \Delta a \leq +|\Delta a|_{max}$ of the plane, obtained in the case of simple transfer, is nothing but the *apparent contour* of the domain accessible in this plane relating to rendezvous, and parallel to axis $\overrightarrow{O\Delta T}$.

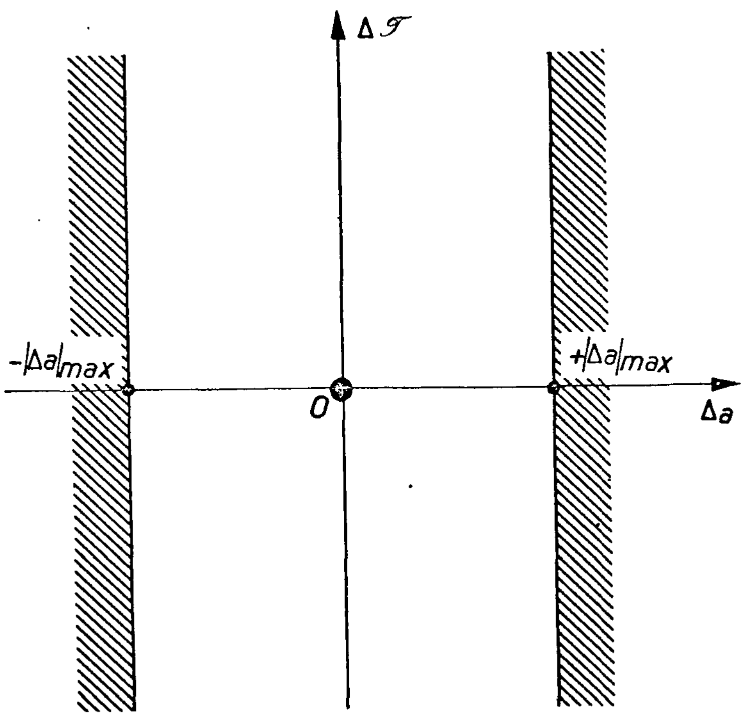

Fig. 5.

Now let us consider the thrust law which produces a maximal variation ΔT (Figure 6). A certain variation Δa corresponds to it.

Equation (6) shows that this thrust law is necessarily such that every acceleration ($da > 0$) is made in the first turn ($k = 0$) and every deceleration ($da < 0$) in the last turn ($k = N$). (Solution of type $A_0 D_H$).

Let us show that the same value ΔT_{max} can be obtained for $\Delta a = |\Delta a|_{max}$. For this it is enough to modify the thrust law in the following way:

Every deceleration at the point of straight ascent L, due to a thrust in direction X, Y, Z, produced in the final turn, is replaced by an acceleration of the same magnitude at the point of straight ascent L + π, due to a thrust in the direction -X, -Y, -Z, produced in the first turn. /167

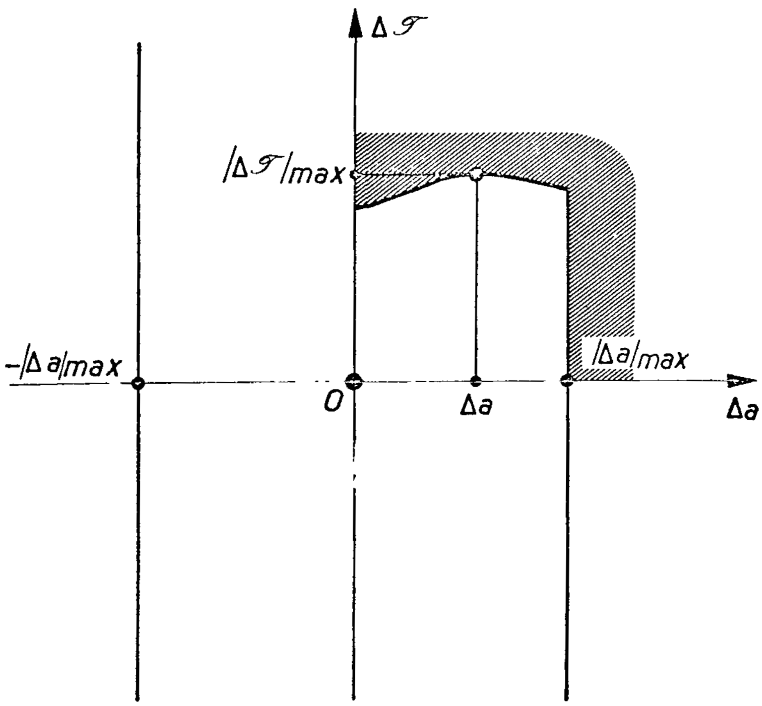

Fig. 6.

This modification does not change $\Delta e_{//}$, $\Delta e_{\perp}$, Δj and ΔT_{max} (nor obviously ΔC).

The new solution, optimal since it leads to ΔT_{max}, and including only first turn accelerations (type A_0), is such that:

$$\Delta \mathcal{T}_{max} = \Delta a \leqslant \Delta a_{max}. \qquad (8)$$

Now, the optimal solution of type I (or, strictly speaking, the boundary between types I and II or I and III) found in the study of simple transfer and leading to Δa_{max} can produce:

$$\Delta \mathcal{T} = \Delta a_{max} \tag{9}$$

on the condition that every acceleration is put into the first turn (bi-impulsional solution of the type $A_o A_o$).

A comparison of equations (8) and (9) shows that $\Delta T_{max} = \Delta a_{max}$ and that the optimal solution $\mathcal{A}_o \equiv A_o A_o$ is of type I.

Let us observe that, for $\Delta T = \Delta T_{max}$, any intermediate value of Δa between $-|\Delta a|_{max}$ and $+|\Delta a|_{max}$ is obtained by a linear combination of $\mathcal{A}_o \mathcal{D}_H$ of the solution of type I: $\mathcal{A}_o \equiv A_o A_o$ (two accelerations in the first turn) leading to $+|\Delta a|_{max}$ and of the solution of type I: $\mathcal{D}_N \equiv D_N D_N$ (two decelerations in the last turn), leading to $-|\Delta a|_{max}$ (Figure 7).

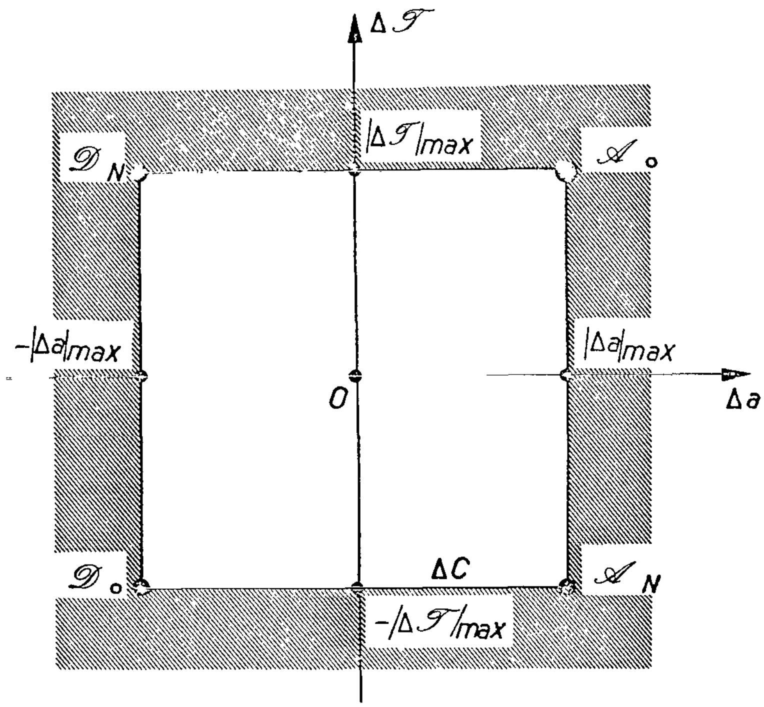

Fig. 7. Domain Accessible in the Plane Δa, ΔT.

The corresponding optimal solution, of a new type (type IV), corresponds to a value of p_τ which is not 0. Its existence is easily deduced from a consideration of the efficiency curve (Figure 8).

In fact, for $p_\tau \neq 0$, this curve can be considered as described by point P, describing with period 2π an ellipse, contained in a fixed plane (π) passing through $\vec{\mathcal{M}Y}$, and shifting slowly and parallel to $\vec{\mathcal{M}Y}$, with the shift per turn (or "pace") being equal to

$$\Delta Y = 3p_\tau\,(2\pi) = \text{order } \frac{1}{\Delta L} \ll 1.$$

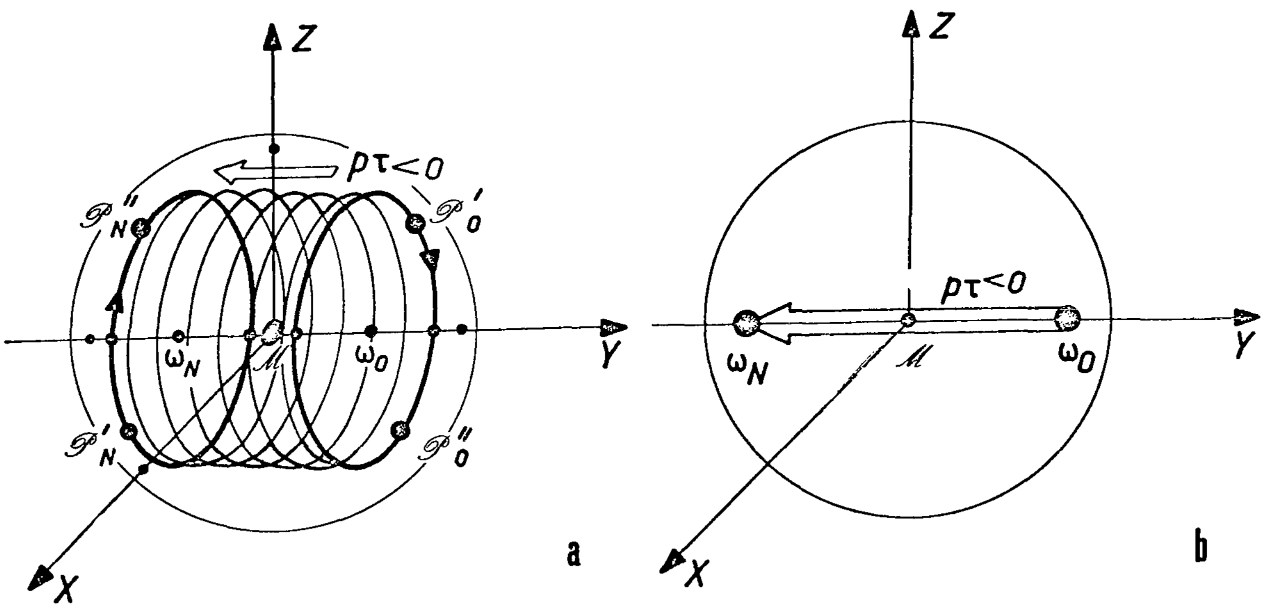

Fig. 8. Efficiency Curve in the Case of Distant Rendez-vous.
a - Type IV
b - Type IV bis

A solution of type IV is quadri-impulsional and can be regarded as a combination of two solutions of type I: one, A_O, in the first turn (two accel- /169 erating impulses $A_O A_O$ at the points of direct ascent L'_O and L''_O, with $X''_O = -X'_O$, $Y''_O = Y'_O$, $Z''_O = -Z'_O$); the other, D_N, in the last turn (two deceleration impulses $D_N D_N$ at the points diametrically opposed to direct ascent $L'_N = L'_O + \pi$ and

$$L''_N = L''_O + \pi, \text{ with } X''_N = -X'_N = X'_O,\ Y''_N = Y'_N = -Y'_O,\ Z''_N = -Z'_N = Z'_O.$$

Let us also note that since point $A_o(|\Delta a|_{max}, |\Delta T|_{max})$ can be obtained with the characteristic velocity ΔC, point $A_N(|\Delta a|_{max}, -|\Delta T|_{max})$ symmetrical to the first in reference to axis $\overrightarrow{O\Delta a}$ and corresponding to a solution of type I with two accelerating impulses $A_N A_N$ applied in the last turn, can also be obtained with this same characteristic velocity, as can every point A of segment $A_o A_N$ (e.g. by a linear combination of the solutions $A_o A_N$; but we shall see that this is not the only possible procedure).

In conclusion, the domain accessible in the plane Δa, ΔT with the characteristic velocity ΔC and for given $\Delta e_{//}$, $\Delta e_{\perp}$, Δj is the inside of the square $A_o D_N D_o A_N$ represented in Figure 7. From this it is deduced that:

section $\Delta a = C^{te}$, $\Delta T = C^{te}$ *of the domain accessible in the three dimensional space* $\Delta e_{//}$, $\Delta e_{\perp}$, Δj *is the same as that corresponding to simple transfer on condition that* $|\Delta a|$ *is replaced by* $\max(|\Delta a|, |\Delta T|)$.

According to the expression: $\max(|\Delta a|, |\Delta T|) = \left\{ \begin{vmatrix} \Delta a \\ \Delta T \end{vmatrix} \right.$, the part of the domain called (I) in the case of simple transfer should be named $\left\{ \begin{matrix} (I) \\ (IV) \end{matrix} \right.$ (Figure4) (or else $\left\{ \begin{matrix} (I\ bis) \\ (IV\ bis) \end{matrix} \right.$, when $\Delta j = 0$; in this last case $|\Delta a|_{max} = 2\Delta C$)

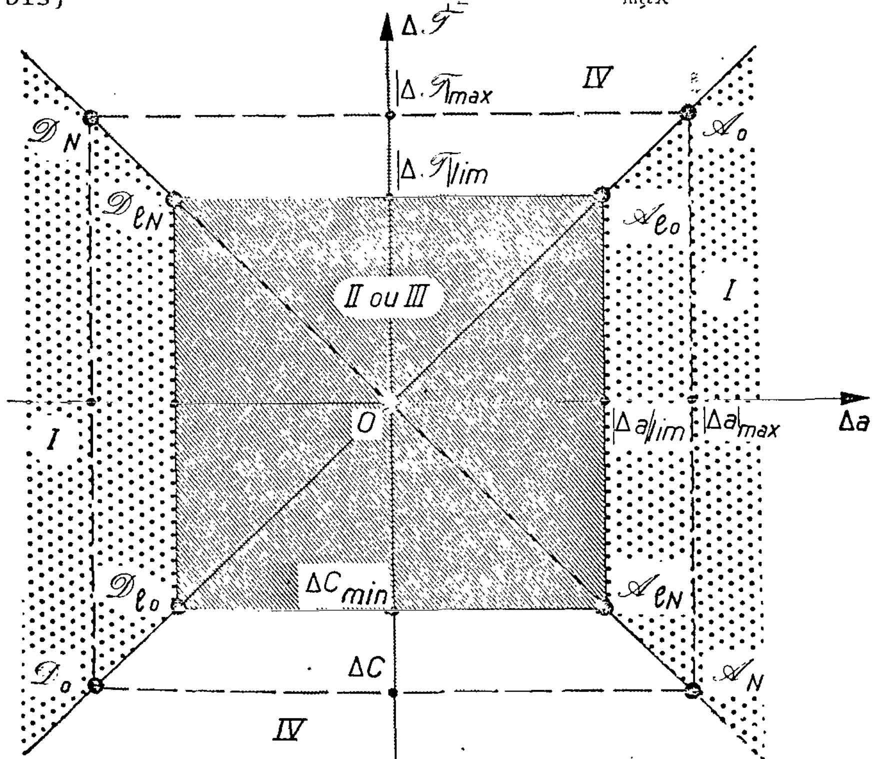

Fig. 9. Iso Lines ΔC.

Now let us trace the iso lines ΔC (Figure 9) in plane Δa, ΔT for fixed $\Delta e_{//}$, $\Delta e_{\perp}$, Δj.

If $\Delta C < \Delta C_{min}$, no rendezvous can be achieved. /170

If $\Delta C = \Delta C_{min}$ (Figures 2 and 3), all points inside the square A_{lo}, D_{lN}, D_{lo}, A_{lN} can be attained by using solutions of the type $\left\{ \begin{matrix} II \\ III \end{matrix} \right.$ depending on whether $3\Delta e_{\perp}^2 \lesseqgtr \Delta j^2$.

If $\Delta C > \Delta C_{min}$, the iso ΔC is formed by the four sides of the square $A_o D_N D_o A_N$.

Figure 10 indicates the values p_{τ} corresponding to each of the regions of plane Δa, ΔT.

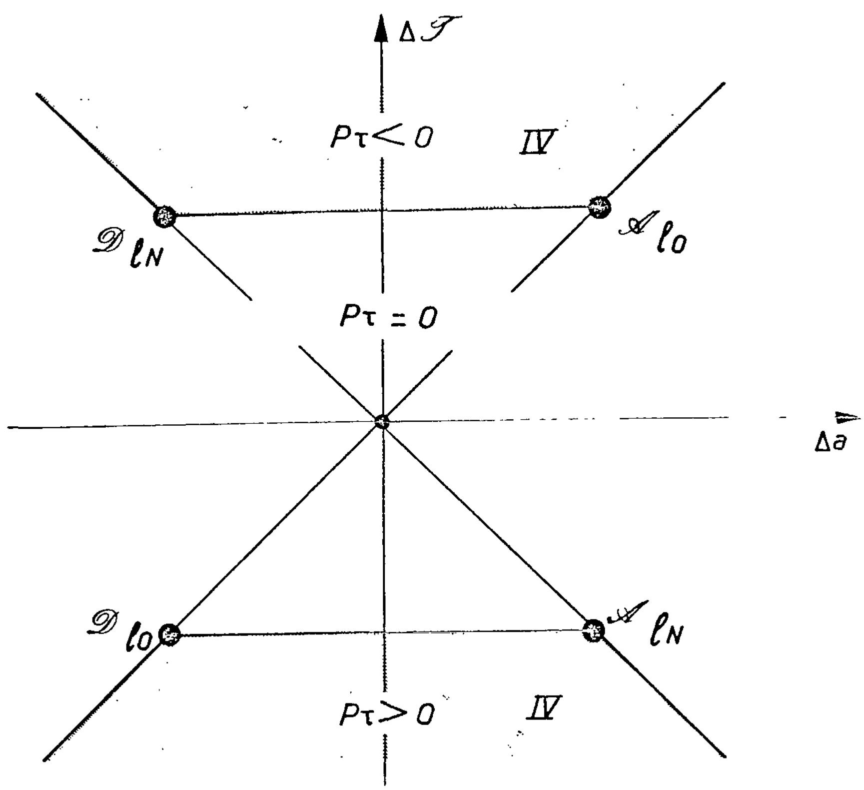

Fig. 10. Value of $p\tau$.

II,6.3. DEGENERATION OF THE OPTIMAL LINEARIZED SOLUTIONS. SOLUTIONS WITH A MINIMAL NUMBER OF IMPULSES.

In the majority of cases the optimal linearized solution is not unique but degenerates into a large number of solutions corresponding to the same characteristic velocity ΔC.

The degeneracy may concern either the thrust magnitude (magnitude degeneracy) alone, or the point of application and the magnitude (spatial degeneracy), or the number of the turn where the thrust is applied and possibly its magnitude (temporal degeneracy).

These cases can also show up simultaneously, which leads to a complex spatio-temporal degeneracy.

Let us recall that, in the case of simple transfers, there is complete temporal degeneracy, since the optimal thrust can be applied to any turn at all with possible fractionalization. There is also spatial degeneracy in the two singular cases of type I bis and III, and magnitude degeneracy for solutions of type II corresponding to $\Delta a = \Delta e_{//} = 0$ [types (9), (10), (11), of Figure II,5 - 11a].

These degenerate cases exist likewise for the rendezvous. /171

There are two types of temporal degeneracies:

The first kind, which will be ignored, is the following: a thrust at the k^{th} turn (different from the first or last) can be partially or completely decomposed into several thrusts applied at the same place and in the same direction, before and after this turn. For example, an impulse ΔV at the k^{th} turn can be decomposed into three impulses $\Delta V/3$ applied respectively to the $(k - 1)^{th}$, k^{th} and $(k + 1)^{th}$ turn.

The second type concerns cases where there are several applications of thrust per turn. It is possible to assign a turn number to each of these applications, since the numbers of turns obtained in this way are found by a linear relationship, which yields a certain possibility of choice.

There also exists spatial degeneracies and magnitude degeneracies.

Let us give a few examples of these degeneracies by giving our attention to *solutions with a minimal number of impulses* (Figures 11, 12, 13), very important in practice, which will also permit us to avoid the study of spatio-temporal degeneracies which are too complex.

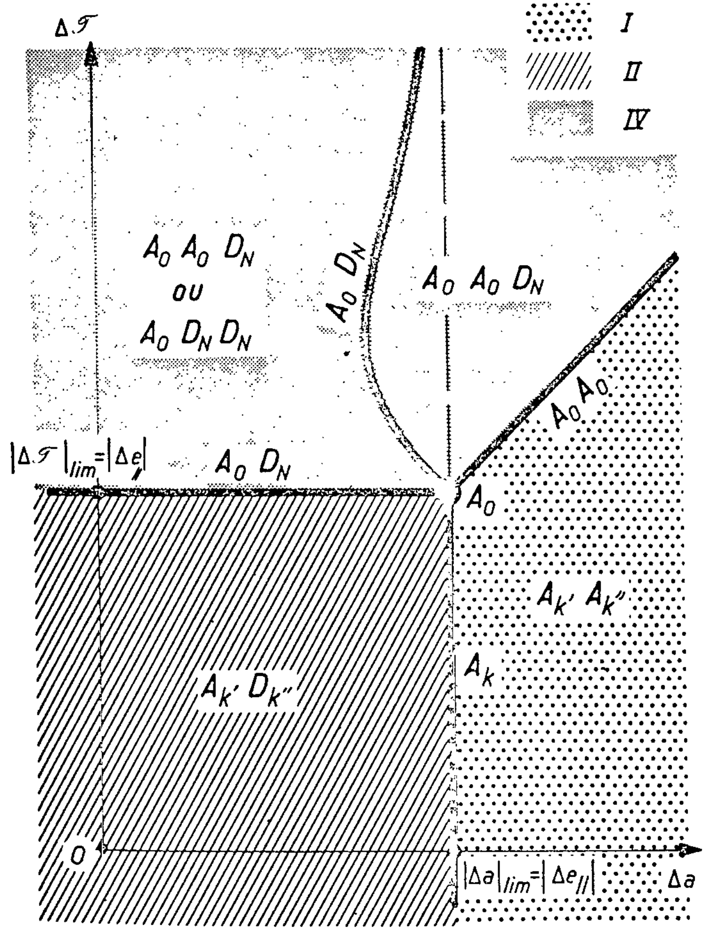

Fig. 11. Minimal Number of Impulses ($|\Delta j| > \sqrt{3}\,|\Delta e_{\perp}|$).

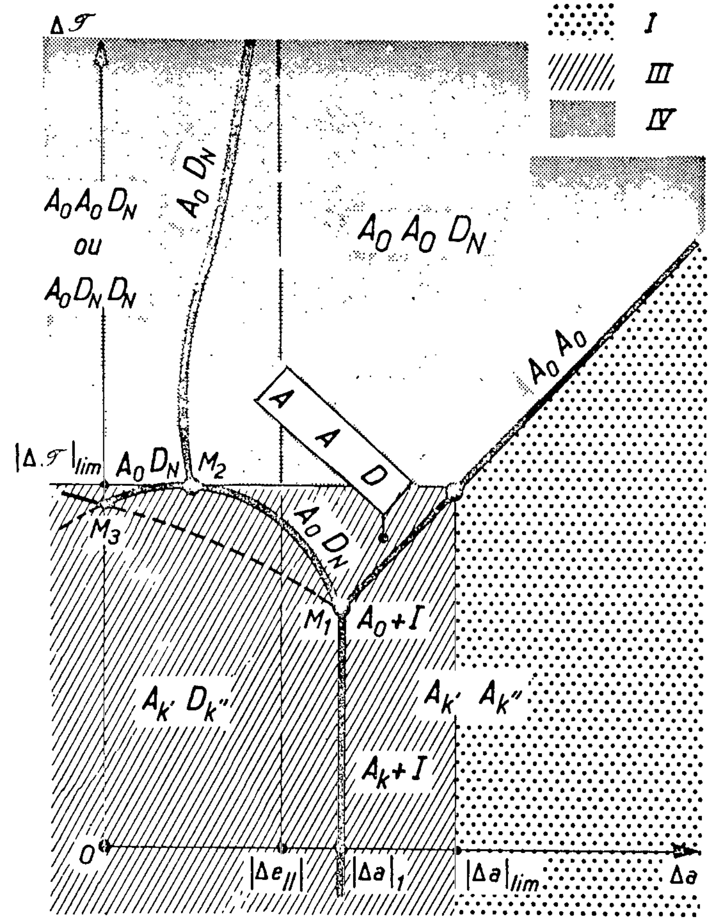

Fig. 12. Minimal Number of Impulses ($|\Delta j| < \sqrt{3}\,|\Delta e_{\perp}|$).

II,6.3.1. Type I. /173

The temporal degeneracy of the first type being removed, the solutions are bi-impulsional:

Impulse I'ΔC, applied to the k'th turn and impulse I''ΔC applied to the k''th turn (solution A_k, $A_{k''}$, if Δa > 0 or D_k, $D_{k''}$ if Δa < 0), with I' + I'' = 1, produce the following variations among others:

$$\Delta a = 2 Y \Delta C$$

$$\Delta \mathcal{T} = 2 Y \Delta C \left[\left(1 - 2 \frac{k'}{N}\right) I' + \left(1 - 2 \frac{k''}{N}\right) I'' \right]$$

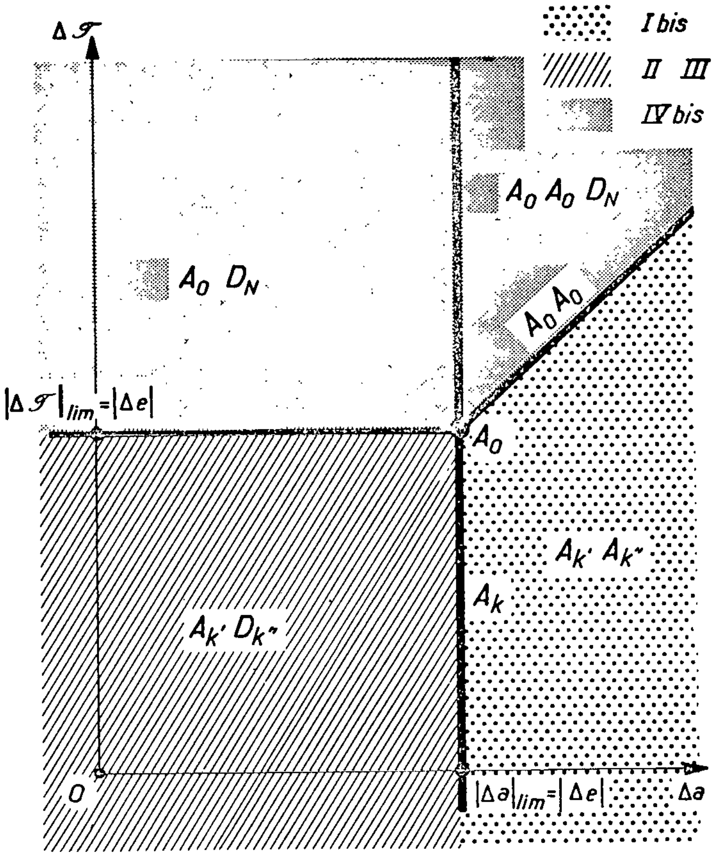

Fig. 13. Minimal Number of Impulses ($|\Delta j| \to 0$).

whence:

$$\boxed{k' I' + k'' I'' = \frac{N}{2}\left(1 - \frac{\Delta \mathcal{T}}{\Delta a}\right)} \qquad (10)$$

For a given rendezvous (I', I'' Δa, ΔT, N fixed), point K of coordinates k' and k'' can therefore be chosen at liberty on segment PQ (Figure 14). A degree of liberty remains. There is temporal degeneracy (of the second type). Every point M of the region I of the plane Δa, ΔT can therefore be generally obtained by an infinity of bi-impulsional solutions. The number of turns referring to these two very well determined impulses are connected by a linear relationship.

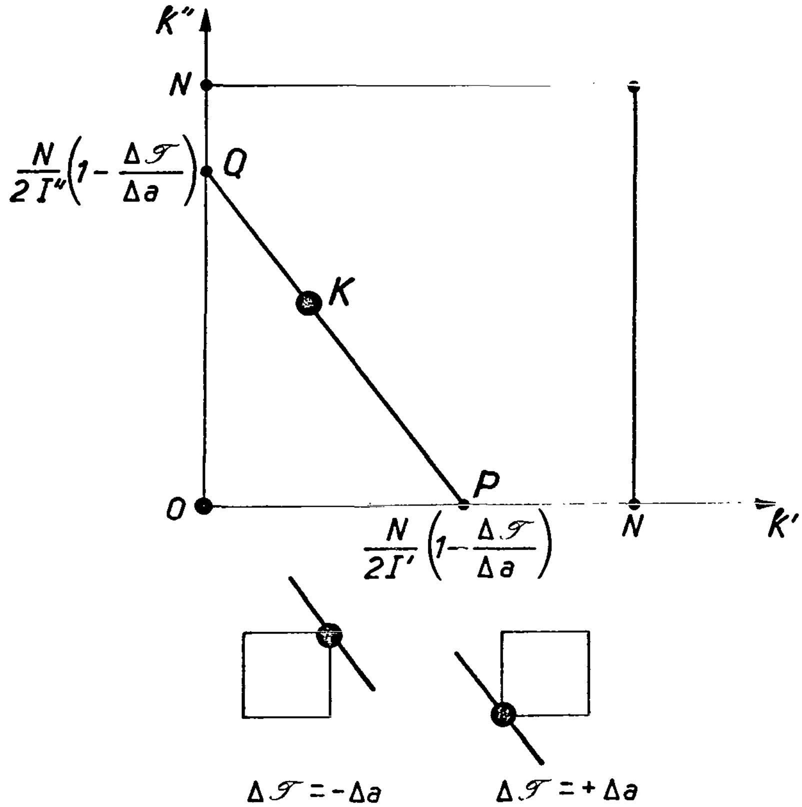

Fig. 14. Temporal Degeneracy of the Bi-impulsional Solutions of Type I.

II,6.3.2. Type II. /174

Analogous results are obtained, this time with a solution of types A_k, $D_{k''}$ including one acceleration and one deceleration. Equation (10) is replaced by:

$$k'I' - k''I'' = \frac{N}{2}(I' - I'')\left(1 - \frac{\Delta\tau}{\Delta a}\right)$$

or again by:

$$\boxed{k'(\Delta a + \Delta e_{/\!/}) + k''(\Delta a - \Delta e_{/\!/}) = N(\Delta a - \Delta\tau)} \tag{11}$$

(See Figure 15).

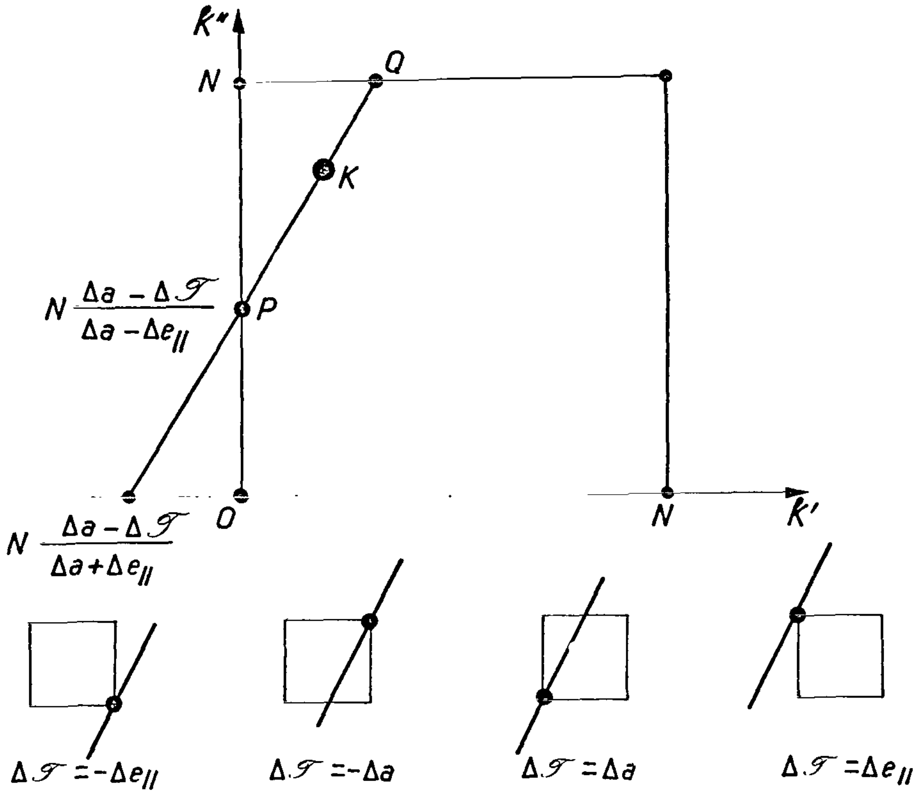

Fig. 15. Temporal Degeneracy of the Bi-impulsional Solutions of Type II.

II,6.3.3. Type III.

In this case, there is a complex spatio-temporal degeneracy.

Let us see under what circumstances the solution can include only two impulses.

The variations $\Delta e_{//}$, $\Delta e_{\perp}$, Δj and consumption ΔC being given, point G representing the transfer associated with the rendezvous under consideration is situated on a parallel (Δ) to axis $\vec{Oz}_1$ (Figure 17) in the axes $\vec{Ox}_1$, $\vec{Oy}_1$, $\vec{Oz}_1$ of Viviani's Window (V) (Figure 16).

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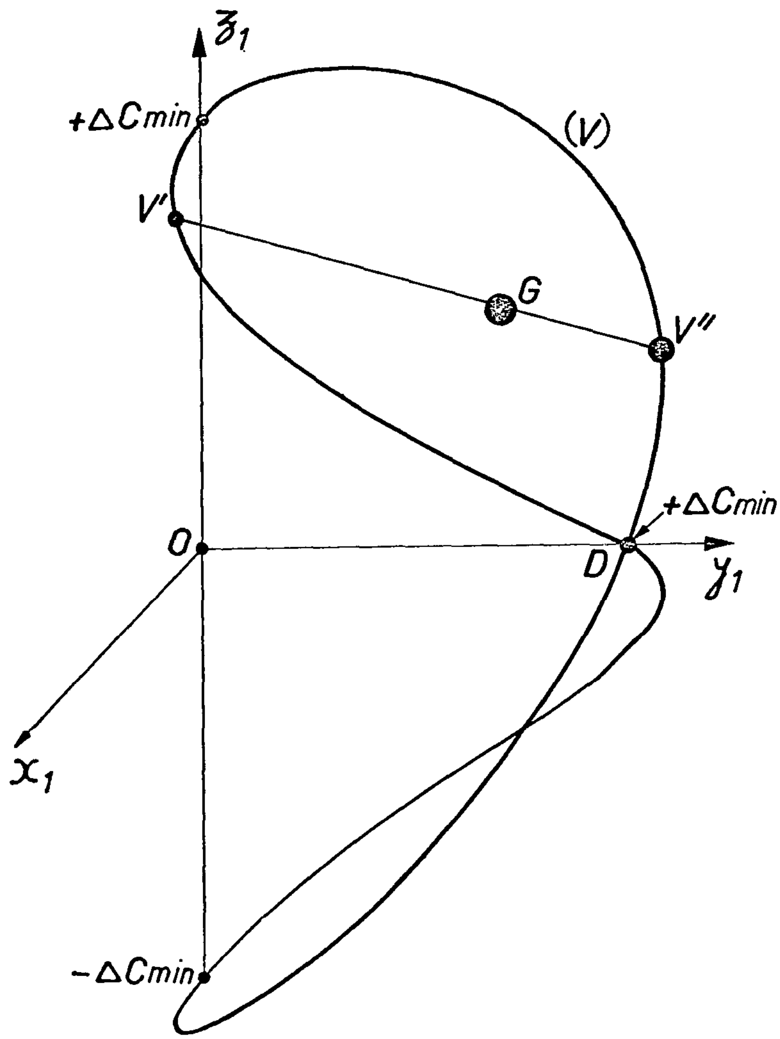

Fig. 16. Type III. Bi-Impulsional Solutions.

Let V'V'' be one of the *two* cords of (*V*) passing through G. To it corresponds the bi-impulsional solution I'ΔC, I''ΔC(I' + I'' = 1) so that: /176

$$\frac{\Delta a}{2} = z_{1G} = I'z_1' + I''z_1'' \tag{12}$$

$$\frac{\Delta \mathcal{T}}{2} = \left(1 - 2\,\frac{k'}{N}\right) I'z_1' + \left(1 - 2\,\frac{k''}{N}\right) I''z_1''. \tag{13}$$

In these equations I', I'', Z'_1 and Z''_1 are fixed by the transfer to be achieved.

Let us indeed fix point G by the datum of its cylindrical coordinates

$$\alpha, r < \frac{\Delta C}{2}, \quad z_1 = \frac{\Delta a}{2}$$

in the axes of center σ (Figure 18). Then:

$$\cos\beta = 2\frac{r}{\Delta C}\cos\varphi \tag{14}$$

$$\begin{cases} \alpha' = 2L' = \alpha + \varphi - \beta \\ \alpha'' = 2L'' = \alpha + \varphi + \beta \end{cases} \tag{15}$$

$$\begin{cases} z_1' = \Delta C \cos L' \\ z_1'' = \Delta C \cos L'' \end{cases} \tag{16}$$

$$\begin{cases} I' = \dfrac{1}{2} + \dfrac{r}{\Delta C}\,\dfrac{\sin\varphi}{\sin\beta} \\ I'' = \dfrac{1}{2} - \dfrac{r}{\Delta C}\,\dfrac{\sin\varphi}{\sin\beta} \end{cases} \tag{17}$$

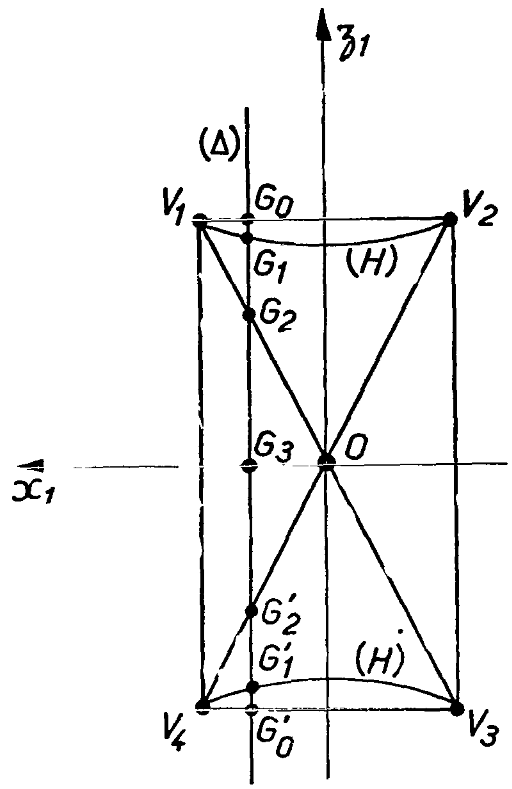

Fig. 17. Section y_1 = constant.

Since the straight line (Δ) is fixed by the datum of α and r, the parameters I', I'', Z_1', Z_1'' depend on the single angular parameter $\phi = (\vec{\sigma G}, \vec{\sigma X})$, which can be theoretically calculated as a function of Δa by (12) (two real solutions corresponding to the two cords V'V'' passing through G).

Then the results are analogous to those obtained in the study of the bi-impulsional solutions of Type I and II, with the equations (10) and (11) being replaced this time, *for each of the two bi-impulsional solutions referring to point* G, by:

$$k'I'z_1' + k''I''z_1'' = \frac{N}{2}\left(I'z_1' + I''z_1''\right)\left(1 - \frac{\Delta \mathcal{T}}{\Delta a}\right) \tag{18}$$

In regard to the maximal value $\Delta T_{max,2}$ of ΔT which it is possible to obtain with Δa being fixed, *with the aid of only two impulses*, two cases are to be envisaged according to whether the points V' and V'' are both situated in the same semi-space $z_1 = \frac{\Delta a}{2} > 0$ [solutions AA, with two accelerating impulses (Figure 19)] or $z_1 < 0$ (solutions DD, with two decelerating impulses), or in two different semi-spaces (solutions AD with one accelerating and one decelerating impulse, Figure 20). /177

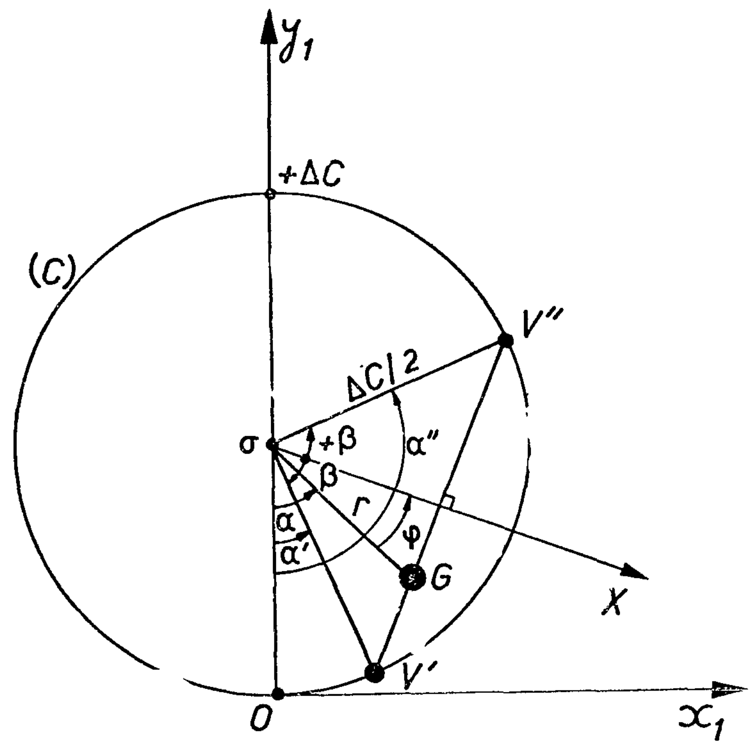

Fig. 18. Notations.

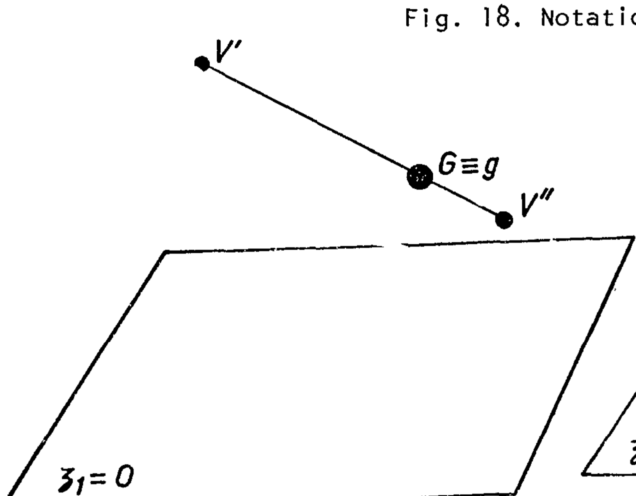

Fig. 19. Two Accelerating Impulses

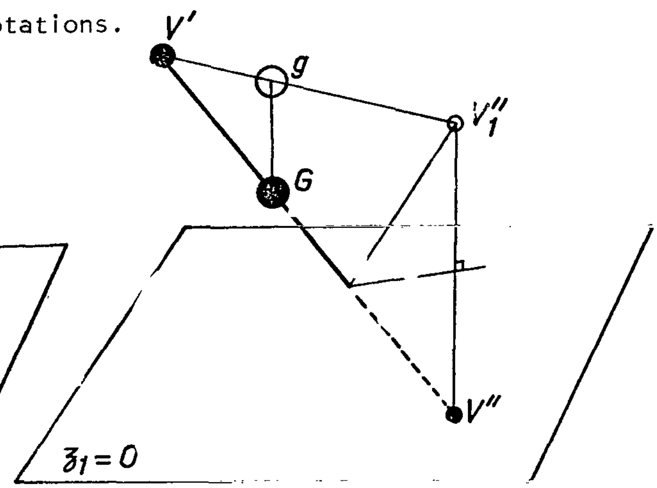

Fig. 20. One Accelerating and One Decelerating Impulse.

The frontier separating the points G corresponding to each of these cases is obtained when V' (or V'') coincides with the double point D of (V). /178

Therefore this frontier is the part of the surface of the circular cone $(\Gamma)[x_1^2 + (y_1 - \Delta C_{min})^2 - z_1^2 = 0]$ of the peak D based on Viviani's Window (V), inside the volume ($\mathcal{V}$) limited by the smallest convex contour [portions of cylinders (C_I) and (C_{II})] fitting (V).

The solutions referring to the *two* cords V'V'' and (V) passing through G belonging to the same case.

When point G shifts onto the straight line (Δ) from G_o to G_3 (Figure 17), the two cords V'V'' passing through G evolve in the following manner:

When G is in G_o, they coincide with the degeneratrix V_1V_2 of the cylinder (C_I).

When G is between G_o and G_1 [where G_1 is one of the two points of intersection G_1 and G_1', of the cone (Γ) and of the straight line (Δ)], they are both situated in the semi-space $z_1 > 0$ (case 1).

When G is in G_1, they coincide with the generatrix DG_1 of the cone (Γ).

When G is between G_1 and G_3, they both cross the plane $z_1 = 0$. (Case 2).

When G is in G_3, they are symmetrical in relation to plane $z_1 = 0$.

Let us study cases 1 and 2 in order.

1st case: two accelerating impulses AA *(or two decelerating impulses* DD*).*

Point G is situated in the volume represented in Figure 21 or, more exactly, on the segment G_oG_1 (or $G_o'G_1'$) of Figure 17. Whence the limitation:

$$\left|\Delta a_1\right| = \sqrt{\Delta e_{/\!/}^2 + \left(\left|\Delta e_\perp\right| - \frac{|\Delta j|}{\sqrt{3}}\right)^2} \leqslant \left|\Delta a\right| \leqslant \left|\Delta a\right|_{lim}. \qquad (19)$$

Let us suppose that $\Delta a > 0$. As in the study of the bi-impulsional solutions of type I, the maximal value of ΔT is obtained when the two accelerating impulses are applied in the first turn (solution A_oA_o). Equation (13) then shows that $\Delta T_{max,2} = \Delta a$ *for each of the two bi-impulsional solutions referring to point* G (Figure 23).

2nd case: one accelerating impulse A *and one decelerating impulse* D.

Point G is situated in the volume represented in Figure 22 or, more exactly, on the segment G_1, G_1' of Figure 17, whence the limitation:

$$\left|\Delta a\right| \leqslant \left|\Delta a_1\right| = \sqrt{\Delta e_{//}^2 + \left(\left|\Delta e_\perp\right| - \frac{|\Delta j|}{3}\right)^2}\ . \tag{20}$$

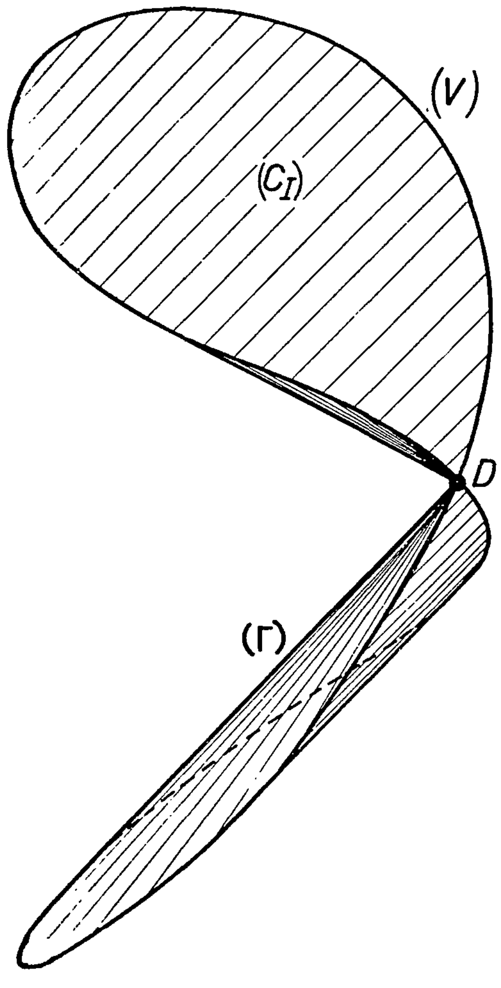

Fig. 21. Points Able to be Reached with the Aid of Two Accelerating Impulses. (or Two Decelerating Impulses).

As in the study of the bi-impulsional solutions of type II, the maximal value of ΔT is obtained when the accelerating impulse I'ΔC is applied in the first turn and the decelerating impulse I''ΔC in the last turn. (Solution $A_o D_H$).

For each of the bi-impulsional solutions referring to point G, equations (12) and (13) in which k' = 0 and k'' = N and $\Delta T = \Delta T_{max,2}$, furnish a *parametric representation as a function of angle* ϕ of the frontier $\overset{\frown}{M_1 M_3}$, (Figure 23) separating the solutions of type III with two impulses (type $A_o D_H$) or more from solutions of type III with three impulses (AAD or ADD) or more.

When in Figure 17 G is in G_1, the two cords V'V" passing through G coincide with the generatrix DG_1 of cone (Γ), the two bi-impulsional solutions referring to G are identical (point M_1 of Figure 23).

When G is in G_2, *one* of the two cords V'V'' passing through G is the diagonal $V_1 V_3$. For the corresponding bi-impulsional solution $\Delta T_{max,2} = \Delta T_{lim}$ (actually point g defined in Figure 20 is then in G_o) and

$$\left|\Delta a\right| = \left|\Delta a_2\right| = 2\left|\Delta e_{//}\right|\left(\frac{|\Delta j|}{3|\Delta j| + \sqrt{3}\,|\Delta e_\perp|}\right)^{1/2} < \left|\Delta e_{//}\right| . \tag{21}$$ /181

A more unfavorable bi-impulsional solution ($\Delta T_{max,2} < \Delta T_{lim}$), corresponds to the other cord.

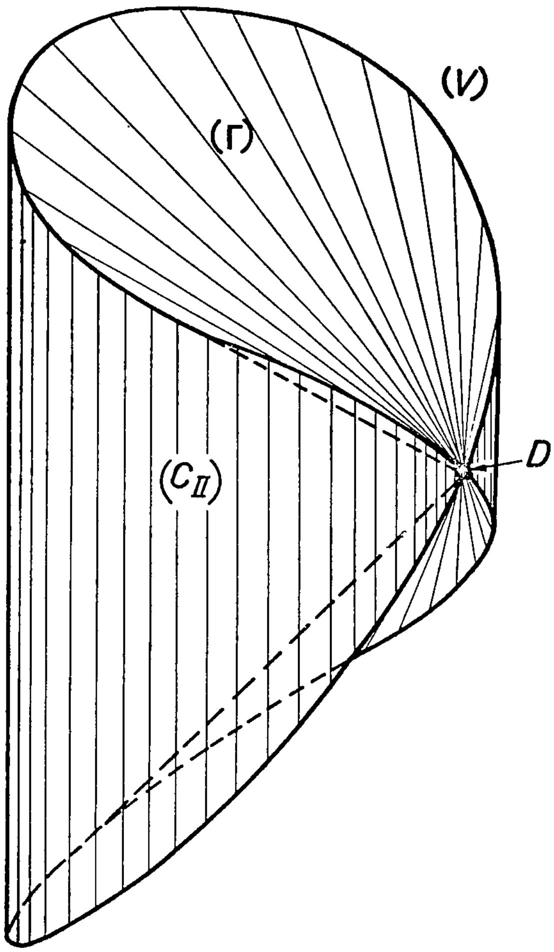

Fig. 22. Points Able to be Reached With the Help of One Accelerating Impulse and One Decelerating Impulse.

When G is in G_3, the two cords passing through G are symmetrical in relationship to the plane $z_1 = 0$. The corresponding bi-impulsional solutions give the same point M_3.

In conclusion, to each point of the region $OKM_1M'M_3O$, there corresponds a double infinity of bi-impulsional solutions of type A_k, $D_{k''}$. In the region $M_3M'M_1MM_2M_3$ only one simple infinity of these solutions corresponds to it.

PARTICULAR CASES.

1. *If* G *is on the cylinder* (C_{II}) ($r = \frac{\Delta C}{2}$)

the two frontiers $M_oM_1MM_2M_3$ and $M_oM_1M'M_3$ coincide with the segment M_oM_4 (limiting case with the solutions of type II).

2. *If* G *is in* σ, center of circle (C), ($r = 0$)

these two frontiers are composed of segments M_oM_1 and of the circular arc $\widehat{M_1M_2}$ (Figure 24).

II,6.3.4. Type IV.

Now let us take a look at case . $\Delta T > 0$.

The accelerating impulses $I_o'\Delta C$ and $I_o''\Delta C$ applied in the first turn and the decelerating impulses $I_N'\Delta C$, $I_N''\Delta C$ applied in the last turn are such that:

$$I_o' + I_o'' - \left(I_N' + I_N''\right) = \frac{\Delta a}{2\gamma\Delta C} \quad \text{fixed} \tag{22}$$

$$I'_0 + I''_0 + I'_N + I''_N = \frac{\Delta\mathcal{T}}{2Y\Delta C} = I' + I'' = 1 \tag{23}$$

$$I'_0 + I'_N - \left(I''_0 + I''_N\right) = I' - I'' = \frac{\Delta e_{/\!/}}{\Delta\mathcal{T}}\,\frac{|\sin L|}{\sin\delta} \quad \text{fixed} \tag{24}$$

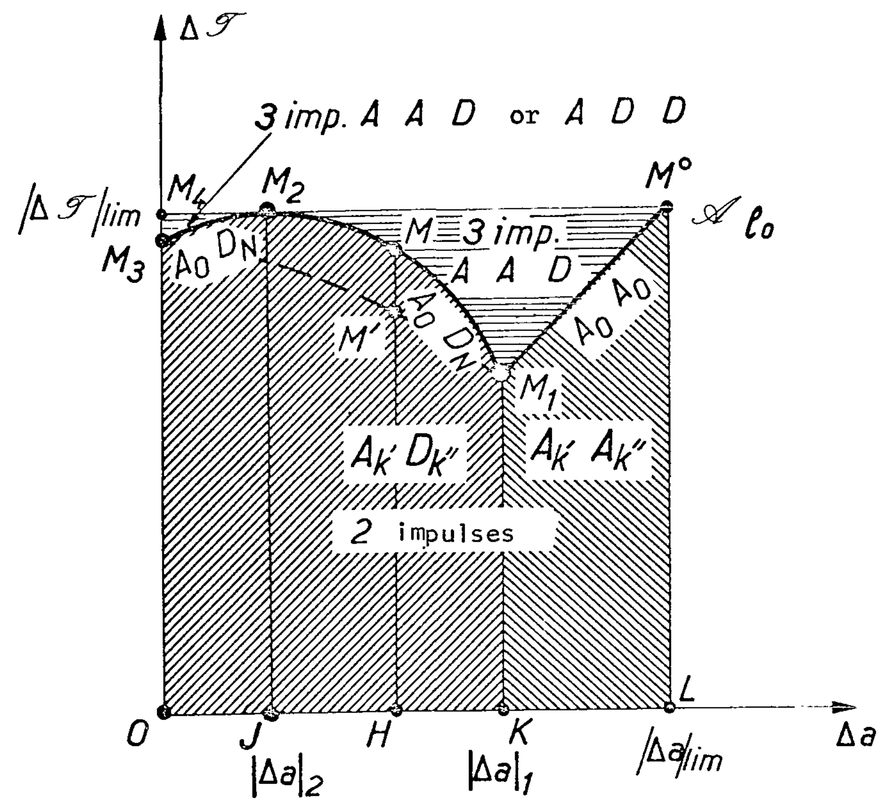

Fig. 23. Type III. Bi-impulsional Solutions.

From these two last equations are derived:

$$I'_0 + I'_N = I' \quad \text{fixed} \tag{25}$$

$$I''_0 + I''_N = I'' \quad \text{fixed} \tag{26}$$

where I' and I'' are the impulse magnitudes corresponding to a simple transfer of type I *where* Δa *is replaced by* ΔT.

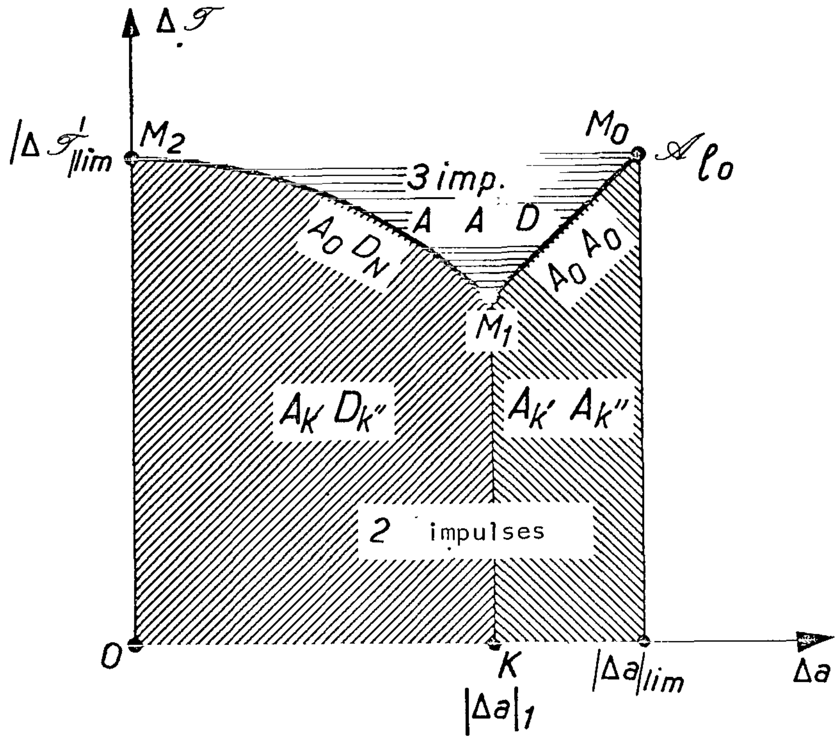

Fig. 24. Type III. Bi-impulsional Solutions. Case r = 0.

Equation (22) then shows that $I'_O + I''_O$ and $I'_N + I''_N$ are constants.

The point $\mathring{I}$ of plane I'_O, I'_N (Figure 25) can then be chosen on the fixed segment JK (J and K are on the periphery OBCE of the rectangle). There is magnitude degeneracy.

If segment JK is in triangle OBG (or else in CEF), using point J or K permits the quadri-impulsional solution of type IV to be reduced to a *tri-impulsional* solution of type $A_O D_N D_N$ (or else $A_O A_O D_N$).

If the segment JK is in the parellogram BGEF, the use of point J leads to /182
a tri-impulsional solution of type $A_O A_O D_N$ and the use of point K to a tri-impulsional solution of type $A_O D_N D_N$.

Finally if the segment JK coincides with BG (or FE), the use of point B (or E) leads to a *bi-impulsional* solution of type $A_O D_N$.

In plane Δa, ΔT let us seek the frontier curve corresponding to these bi-impulsional solutions (Figures 11 and 12).

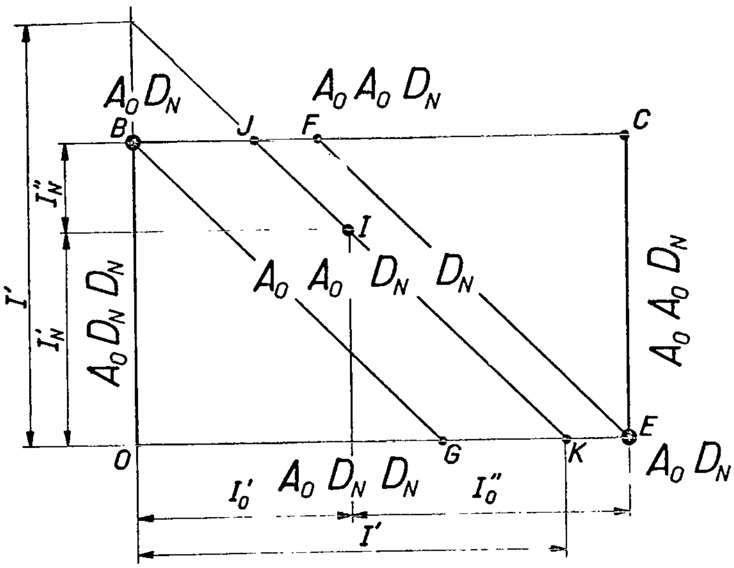

Fig. 25. Type IV. Magnitude Degeneracy.

As $I''_O = I'_N = 0$ at point E, equations (22 - 24) give:

$$\left|\frac{\Delta a}{\Delta \tau}\right| = \left|I' - I''\right| = \left|\frac{\Delta e_{//}}{\Delta \tau}\ \frac{\sin L}{\sin \delta}\right| \tag{27}$$

or, taking into consideration the study of chapter II,5.,

$$\left|\frac{\Delta a}{\Delta e_{//}}\right| = \left|\frac{\sin L}{\sin \delta}\right| = \sqrt{1 - x^2} \leqslant 1 \tag{28}$$

where $x = \dfrac{\sqrt{\Delta T^2 - \Delta e^2_{//}}}{|\Delta e_{//} \mathrm{tg}\delta|}$ is the root of the *smallest* modulus (i.e.here of modulus $\leq$ 1) of the reciprocal second degree equation:

$$x^2 + \frac{\Delta \tau^2 + \Delta j^2 + \Delta e_\perp^2 - \Delta e_{//}^2}{|\Delta e_\perp| \sqrt{\Delta \tau^2 - \Delta e_{//}^2}}\, x + 1 = 0 \tag{29}$$

which depends only on the parameter operating as a coefficient of x. $|\Delta a|$ is minimum

$$\left(|\Delta a|_{min} = \frac{\sqrt{2|\Delta j|\sqrt{\Delta e_\perp^2 + \Delta j^2} - 2\Delta j^2}}{|\Delta e_\perp|}\right)$$

for $\quad \Delta\mathcal{T} = \sqrt{\Delta e^2 + \Delta j^2} \; (\gtrless \Delta\mathcal{T}_{lim}$ $\quad$ depending on whether $\; |\Delta j| \gtrless \frac{|\Delta e_\perp|}{\sqrt{3}})$ /183

When $|\Delta T| \to \infty$, $|\Delta a| \to |\Delta e_{//}|$. Finally, for $|\Delta T| = |\Delta T|_{lim}$ again we find

$$|\Delta a| = \begin{cases} |\Delta e_{//}| \\ |\Delta a_2| \end{cases} \qquad \text{depending on whether } |\Delta j| \gtrless \sqrt{3}\,|\Delta e_\perp|.$$

II,6.3.5. Type I bis.

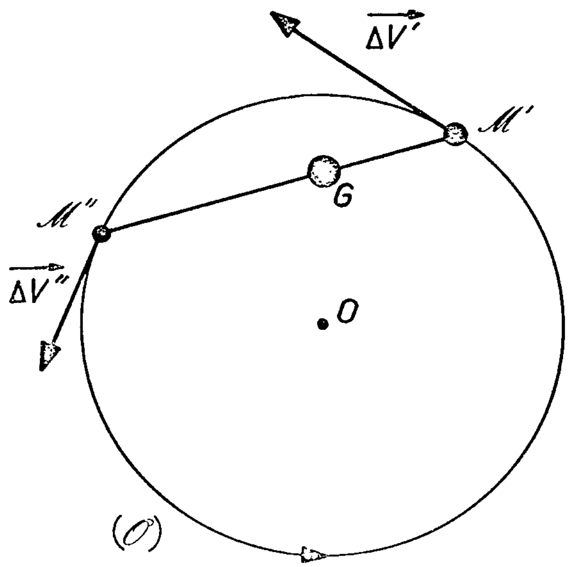

Fig. 26. Type I bis. Bi-impulsional Solutions.

Here again there is a rather complex spatial and temporal degeneracy. Therefore we shall interest ourselves only in the bi-impulsional solutions (Figure 26).

There exists an infinity of bi-impulsional solutions guaranteying the transfer $\Delta e_{//}$, $\Delta e_\perp$, $\Delta j = 0$, $|\Delta a| = 2\Delta C$. The tangential impulses $\Delta V'$ and $\Delta V''$ are such that the center of gravity of masses $\Delta V'$ and $\Delta V''$ placed at application points M' and M'', is in G so that

$$\overrightarrow{OG} = \frac{\overrightarrow{\Delta e}}{\Delta a}. \qquad (30)$$

The datum of any cord $M'M''$ passing through G fixes the relationship $\Delta V''/\Delta V'$ of the impulses and therefore the solution.

There is spatial degeneracy of the bi-impulsional solutions.

Moreover, once a particular bi-impulsional solution has been chosen there exists a temporal degeneracy (of the second type) similar to that found in the study of type I. [Equation (10)].

II,6.3.6. Type IV bis.

Here there is no more temporal degeneracy since, for example if $\Delta T > 0$, all accelerations are made in the first turn (A_0) and all the decelerations in the last turn (D_N).

However there continues to be a spatial degeneracy: the solutions

A_o and D_N are of the degenerated type I bis and the point M of the segment A_oD_N is obtained through a linear combination of these two solutions (Figure 9).

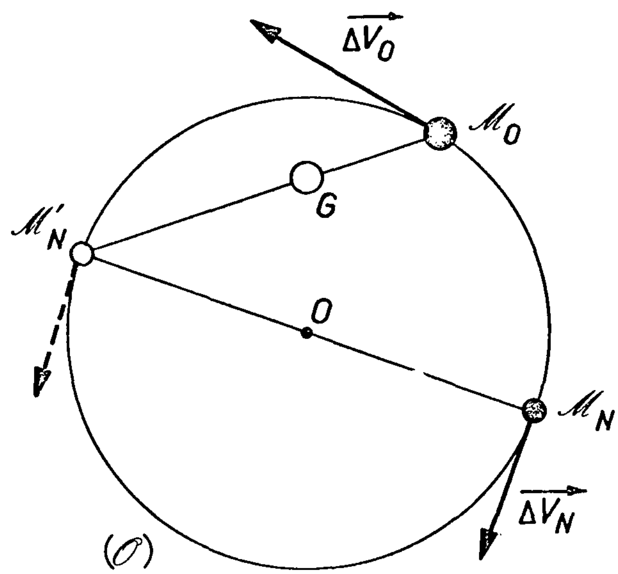

Fig. 27. Type IV bis. Bi-impulsional Solutions.

Since both points A_o and D_N can be obtained in an infinity of ways, in particular by bi-impulsional solutions, point M can be obtained, in particular, and in an infinity of ways by quadri-impulsional solutions or rather, in a simple or double infinity of ways by tri-impulsional solutions, because of the magnitude degeneracy already pointed out for solutions of type IV (Figure 25).

Point M can even, in certain /184
cases, be obtained by bi-impulsional solutions. Actually the accelerating tangential impulse $\overrightarrow{\Delta V}_o$, applied in the first turn, and the decelerating tangential impulse $\overrightarrow{\Delta V}_N$ applied in the last turn (Figure 27), produce the variations:

$$\Delta a = 2(\Delta V_0 - \Delta V_N) \tag{31}$$

$$\Delta \mathcal{T} = 2(\Delta V_0 + \Delta V_N) = 2\Delta C \tag{32}$$

$$\overrightarrow{\Delta e} = 2\Delta V_0\, \overrightarrow{O\mathcal{M}_0} - 2\Delta V_N\, \overrightarrow{O\mathcal{M}_N} = 2\Delta C\, \overrightarrow{OG} \tag{33}$$

where G is the center of gravity of *positive* masses ΔV_o placed on M_o and ΔV_N placed on M'_N, symmetrical with M_N in relationship to 0.

Since Δa and ΔT are fixed, equations (31) and (32) furnish the magnitudes ΔV_o and ΔV_N of the impulses.

Since $\overrightarrow{\Delta e}$ (therefore G) are fixed, this is a matter of constructing the cord(s) $M_oM'_N$ passing through G so that:

$$\frac{G\mathcal{M}'_N}{G\mathcal{M}_0} = \frac{\Delta V_0}{\Delta V_N} = \rho = \text{fixed ratio.}$$

Two cords $M_{o1}M'_{N1}$ and $M_{o2}M'_{N2}$, symmetrical in relationship to OG, answer the question. M'_{N1} and M'_{N2} are the intersection points of circle (O) and circle (C) which is derived from (O) by the similarity of center G and of relationship ρ (Figure 28). These points are only real if

$$\left|1-\rho\right| < O\omega = \frac{|\overrightarrow{\Delta e}|}{2\Delta C}\left(1+\rho\right) < 1+\rho . \qquad (35)$$ /185

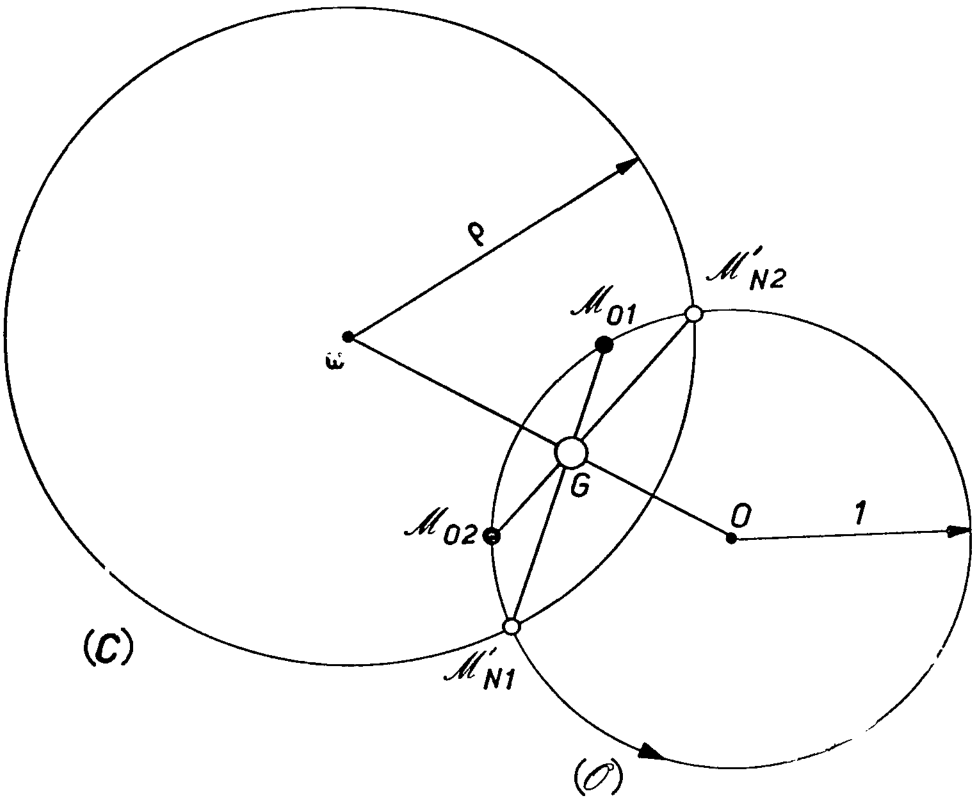

Fig. 28. Type IV bis. Determination of Application Points of the Two Impulses.

The second inequality

$$\left|\overrightarrow{\Delta e}\right| < 2\Delta C$$

is automatically satisfied in zone IV bis for:

$$|\Delta e| = |\Delta \mathcal{T}|_{lim} < |\Delta \mathcal{T}| = 2\Delta C .$$

The first inequality is also written:

$$|\Delta a| < |\overrightarrow{\Delta e}| = |\Delta a|_{lim} . \tag{36}$$

Therefore there are no bi-impulsional solutions (one impulse in the first turn and one impulse in the final turn) except in the belt $|\Delta a| < |\Delta a|_{lim}$ (Figure 13).

II,6.4. CONCLUSION.

The study of the economical long-duration impulsional rendezvous between non-co-planar, close, near-circular orbits is derived simply from the study of the corresponding simple transfers. The accessible domain corresponding to the characteristic velocity ΔC is obtained by replacing $|\Delta a|$ by $\max(|\Delta a|, |\Delta T|)$ in the study of simple transfer with

$$\Delta \mathcal{T} = -\frac{4}{3}\,\frac{\Delta \tau}{\Delta L} + \Delta a .$$

The optimal solution is one of the four types I, I bis, II or III already found in Chapter II,5, or of two new types: types IV and IV bis.

The minimal number of impulses is equal to 3 or 2 according to the rendezvous under consideration.

CONCLUSION

The analytical study of the optimal transfers between close elliptical orbits is a great deal more delicate for propulsion systems (S_1) with a constant ejection velocity and limited thrust than for propulsion systems (S_2) with variable ejection velocity and limited power where the general analytical solutions can be obtained. This is due to the fact that the "all or nothing" optimal thrust law obtained for propulsion systems (S_1) introduces non-linearities even in the linearized study (close orbits). /186

In spite of the simplifications introduced by the choice of orbital elements as state components (which assures a constant adjoint in the linearized study), three essential difficulties continue to exist in the case of propulsion systems (S_1):

1. Determination of the optimal thrust law, i.e. of the succession of thrust arcs and ballistic arcs as a function of the a priori fixed adjoint. This determination is facilitated by the utilization of notions of "efficiency vector" (indicating the direction of optimal thrust), of "directrix orbit" (locus of the extremity of the efficiency vector, originally the mobile, in the absolute axes) and of "efficiency curve" (locus of this extremity in the turning axes).

   There are no more than three maximal thrust arcs (except for a singular case) per revolution for a simple transfer. In the singular cases (of the linearized problem), the optimal thrust law is partially arbitrary (degeneracy of the solution). Such cases are only found in transfers between near-circles (eccentricity of the order of the transfer size). The degeneracy disappears when the study is pursued at higher orders.

2. Integration. Integrating the differential equations producing the variations in orbital elements in consumption, into which the law of optimal thrust has been introduced, is delicate. In general the integrals to be calculated are of the elliptical type or more complicated.

3. Inversion. The a posteriori determination of the adjoint as a function of the transfer data can generally only be made by successive approximations.

   These difficulties explain the rareness of general results which it is possible to obtain.

The phenomena of "induction" (non-imposed variation of certain orbital elements induced by the imposed variations of the other orbital elements) can, however, be studied in detail.

The complete analytical solution can be obtained in a certain number of particular cases, the study of which is complicated as the orbital elements on which the variation is imposed are more numerous. A certain number of simplifying hypotheses, some of which can be cumulative, should be made by a counterpart system (great number of revolutions, near-circular orbits, impulses, etc. ...). Very fortunately the simplifying hypotheses correspond to very frequent practical cases.

The study of the particular cases shows that the non-modulation of the ejection velocity, for propulsion systems (S_1) and in relationship to propulsion systems (S_2) of the same power, entails a penalty on the mass consumption of the order of 20 to 30% if the maximal thrust is applied in continuous fashion (system S_{1c}), and a lesser penalty if the (constant) ejection velocity is optimized (system S_1^*).

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APPENDIX 1

DETERMINATION OF THE OPTIMAL EJECTION VELOCITY FOR A GIVEN MISSION (SYSTEM S_1)

(see § I,2.3.3.)

As it is mf (or, which comes to the same thing $-C_f/W$) and not $-C_f$ /191
which must be maximized, it is necessary to use the equations of movement (I,2 -11) - (I,2 -13) to which must be added:

$$\dot{W} = 0 \tag{1}$$

expressing the fact that the ejection velocity W is constant.

The useful part of the optimal Hamiltonian is written:

$$H^* = \frac{F^*}{m}\left(|\overrightarrow{p_V}| - \frac{m p_m}{W}\right) + \cdots = \frac{\Theta\, U(\Theta) F_{max}(W)}{m} + \cdots \tag{2}$$

where

$$\Theta = |\overrightarrow{p_V}| - \frac{m p_m}{W} \left(= |\overrightarrow{p_V}| + p_c\right) \tag{3}$$

is the commutation function.

It is important to note that the limit:

$$F_{max} = \frac{2 P_{max}}{W} \tag{4}$$

depends on the state (by W).

The adjoint component associated with W is given by:

$$\begin{aligned} \dot{p}_W &= -\frac{\partial H^*}{\partial W} = \frac{\Theta U(\Theta)}{m} \frac{\partial F_{max}}{\partial W} + \frac{F_{max}}{m} \frac{d\left[\Theta U(\Theta)\right]}{d\Theta} \frac{\partial \Theta}{\partial W} = \\ & \frac{\Theta U(\Theta)}{m}\left(-\frac{2P_{max}}{W^2}\right) + \frac{F_{max}}{m} \cdot U(\Theta) \cdot \left(\frac{m p_m}{W^2}\right) = \\ & \frac{2P_{max} U(\Theta)}{m W^2}\left(\frac{m p_m}{W} - \Theta\right) = -\frac{2P_{max} U(\Theta)}{m W^2} (\Theta + p_c) \end{aligned} \tag{5}$$

and

$$p_{W_o} = p_{W_f} = 0 \tag{6}$$

since W is indifferent. /192

By integrating $\dot{p}_W$ from t_o to t_f, it then becomes:

$$\int_{t_o}^{t_f} \frac{U(\Theta)}{m}\left(\Theta + p_c\right) dt = 0 \tag{7}$$

which is nothing but equation (I,2 -43).

APPENDIX 2

PERIGEE VECTOR (see § I,2.4.1.)

A2.1 - LA PLACE'S FIRST INTEGRAL [1] /193

In the central Newtonian field, with center 0 (Figure I,2 - 9), let us calculate the vectoral product $\frac{1}{\mu}\vec{\Gamma} \wedge \vec{h}$ for a Keplerian movement:

$$\frac{1}{\mu}\vec{\Gamma} \wedge \vec{h} = \frac{1}{\mu}\vec{g} \wedge \vec{h} = -\frac{\vec{r}}{r^3} \wedge \vec{h} = -\frac{h}{r^2}\vec{X} \wedge \vec{Z}$$

or

$$\frac{1}{\mu}\vec{\Gamma} \wedge \vec{h} = \dot{\theta}\vec{Y} = \dot{\vec{X}} \tag{1}$$

Whence, by integrating:

$$\boxed{\frac{1}{\mu}\vec{V} \wedge \vec{h} - \frac{\vec{r}}{r} = \text{constant vector} = \vec{e}} \tag{2}$$

that is LaPlace's (vectorial) first integral (three first scalar integrals).

Put into the form:

$$-\vec{Z} \wedge \vec{V} = \mu\frac{\vec{e}}{h} + \mu\frac{\vec{X}}{h} \tag{3}$$

it directly furnishes the hodograph (H') which is deduced from the hodograph (H) by rotating $-\frac{\pi}{2}$ around $\vec{OZ}$. This is a circle with center $\dot{I}$ defined by:

$$\vec{OI} = \mu\frac{\vec{e}}{h} \tag{4}$$

and of radius μ/h (Figure 1).

The trajectory (O) is deduced immediately from this for, in every central acceleration movement, the trajectory (O) and the reversed hodograph (H') are reciprocal polars in respect to circle (σ) of center 0 and of radius $\sqrt{h}$.

Actually:

$$OH.OV'=h = C^{te} \tag{5}$$

which shows that the tangent $\overrightarrow{MV}$ to the trajectory (O) is nothing but the polar of point V' in reference to the circle (σ).

The point H, the inverse of V' in reference to (σ) describes the circle (C), inverse of the circle (H') in reference to (σ). Therefore, the straight line HM encloses the conical (O) of focus O and of principal circle (C).

On the other hand it is easy to demonstrate that:

$$|\vec{e}| = e \tag{6}$$

The constant vector $\vec{e}$ is therefore directed toward the perigee *and has* e *as its length.* We shall call it the "perigee vector". /194

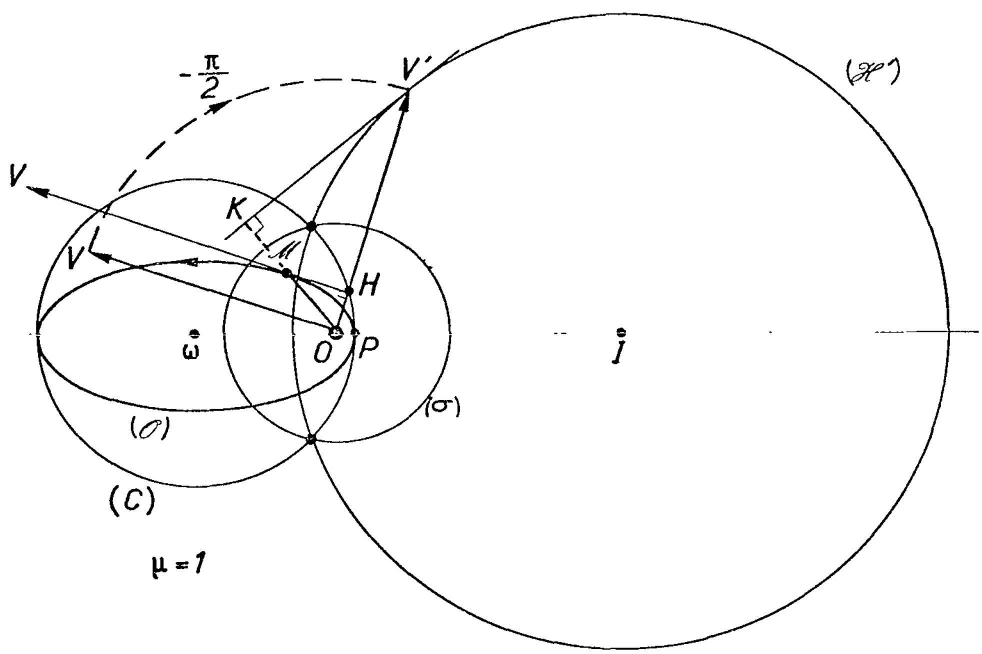

Fig. 1.

A2.2 - PERTURBATION OF THE PERIGEE VECTOR

Let us calculate the derivative of the perigee vector when the mobile M is subjected to the perturbating acceleration $\vec{\gamma}$

$$\dot{\vec{e}} = \frac{1}{\mu}\vec{\Gamma}\wedge\vec{h} + \frac{1}{\mu}\vec{V}\wedge\dot{\vec{h}} - \frac{\dot{\vec{r}}}{r} + \frac{\vec{r}}{r^2}\dot{r}$$

where:

$$\vec{\Gamma} = \vec{g} + \vec{\gamma} = -\mu \frac{\vec{r}}{r^3} + \vec{\gamma}$$

$$\vec{h} = \vec{r} \wedge \vec{V}$$

$$\dot{\vec{h}} = \vec{r} \wedge \vec{\Gamma} = \vec{r} \wedge \vec{\gamma}$$

$$\dot{r} = \vec{V} \cdot \frac{\vec{r}}{r}$$

whence:

$$\dot{\vec{e}} = \underbrace{\left[-\frac{\vec{r}}{r^3} \wedge \left(\vec{r} \wedge \vec{V} \right) \right]} + \frac{1}{\mu} \vec{\gamma} \wedge \vec{h} + \frac{1}{\mu} \vec{V} \wedge \left(\vec{r} \wedge \vec{\gamma} \right) - \frac{\vec{V}}{r} + \frac{\vec{r}}{r^2} \left(\vec{V} \cdot \frac{\vec{r}}{r} \right)$$

$$- \vec{r} \left(\frac{\vec{r}}{r^3} \cdot \vec{V} \right) + \left(\vec{r} \cdot \frac{\vec{r}}{r^3} \right) \vec{V}$$

therefore /195

$$\boxed{\mu \dot{\vec{e}} = \vec{\gamma} \wedge \vec{h} + \vec{V} \wedge \left(\vec{r} \wedge \vec{\gamma} \right)} \qquad (7)$$

When the perturbating acceleration is zero ($\vec{\gamma} = 0$), $\dot{\vec{e}} = 0$. The vector $\vec{e}$ is then very constant for a Keplerian movement.

APPENDIX 3

ELEMENTS OF THE MATRIX K
(see § I,3.1.4)

with: $a = 1$, $n = 1$, $b = \sqrt{1 - e^2}$, $h = \sqrt{1 - e^2}$, $p = 1 - e^2$, $r = 1 - e\cos u$ /197

	$e \neq 0$	$e = 0$
$K_{\xi Z}$	$\frac{r}{h}\sin L = \sin u$	$\sin L$
$K_{\eta Z}$	$-\frac{r}{h}\cos L = -\frac{\cos u - e}{b}$	$-\cos L$
K_{aX}	$\frac{2e}{b}\sin L = \frac{2e}{r}\sin u$	0
K_{aY}	$2\frac{b}{r}$	2
$K_{\alpha X}$	$b\sin L = \frac{p}{r}\sin u$	$\sin L$
$K_{\alpha Y}$	$b(\cos L + \cos u) = \frac{b}{r}(2\cos u - e\cos^2 u - e)$	$2\cos L$
$K_{\beta X}$	$-b\cos L = -\frac{b}{r}(\cos u - e)$	$-\cos L$
$K_{\beta Y}$	$b\sin L + \sin u = \frac{\sin u\,(2 - e\cos u - e^2)}{r}$	$2\sin L$
$K_{\tau X}$	$-2r + \frac{3Me\sin L}{b} = -2r + \frac{3Me\sin u}{r}$	-2
$K_{\tau Y}$	$\frac{3bM}{r}$	$3L$

APPENDIX 4

CANONICAL TRANSFORMATION
(see § I,3.2.1)

The canonical transformation envisaged is: /199

$$\left(\vec{r}, \vec{V}, t\,;\, \vec{p_r}, \vec{p_V}, H\right) \Longrightarrow \begin{cases} \left(\vec{\mathcal{X}}, u\,;\, \vec{P}, \mathcal{H}\right) \\ \qquad\qquad \text{or else} \\ \left(\vec{Z}_{plan}, a, \vec{e}_{plan}, \tau, u\,;\, \vec{p_Z}, p_a, \vec{p_e}, p_\tau, \mathcal{H}\right) \end{cases} \tag{1}$$

retaining the variables m (or C, or J). The sufficient condition for this change of variables to be canonical is written:

$$\begin{aligned} \vec{p_r}\cdot\vec{dr} + \vec{p_V}\cdot\vec{dV} - H dt \equiv \vec{P}\cdot\vec{d\mathcal{X}} - \mathcal{H} du \equiv \vec{p_Z}\cdot\vec{dZ} + p_a\, da\, + \\ \vec{p_e}\cdot\vec{de} + p_\tau\, d\tau - \mathcal{H} du. \end{aligned} \tag{2}$$

In identifying the terms in $\vec{dV}$ for $\vec{dr} = 0$, dt = 0 (*fixed position and time*), we evidently have:

$$\vec{d\mathcal{X}} = \overline{\overline{K}}\cdot\vec{dV} \tag{3}$$

$$\vec{dZ} = \frac{1}{h}\, d\left(\vec{r}\wedge\vec{V}\right) = \frac{\vec{r}}{h}\wedge\vec{dV} \tag{4}$$

$$da = 2\, dE = 2\, d\left(\frac{V^2}{2} - \frac{\mu}{r}\right) = 2\vec{V}\cdot\vec{dV} \tag{5}$$

$$\vec{de} = d\left(\vec{V}\wedge\vec{h} - \frac{\vec{r}}{r}\right) = \vec{dV}\wedge\vec{h} + \vec{V}\wedge\vec{dh} = \vec{dV}\wedge\vec{h} + \vec{V}\wedge\left(\vec{r}\wedge\vec{dV}\right) \tag{6}$$

$$d\tau = \frac{3}{2}\, M\, da + \vec{\mathcal{R}}\cdot\vec{de} \tag{7}$$

whence:

$$\vec{p_V} = \vec{P}\cdot\overline{\overline{K}} = 2 p_{a_1}\vec{V} + \vec{h}\wedge\vec{p_{e_1}} + \left(\vec{p_{e_1}}\wedge\vec{V} + \frac{\vec{p_Z}}{h}\right)\wedge\vec{r} \tag{8}$$

with /200

$$p_{a_1} = p_a + \frac{3M}{2} p_\tau \tag{9}$$

$$\overrightarrow{p_{e_1}} = \overrightarrow{p_e} + \vec{\mathfrak{R}} p_\tau . \tag{10}$$

These equations coincide with equations (I,3.41-43).

APPENDIX 5

ELEMENTS OF THE NOMINAL DIRECTRIX ORBIT
(see § I,3.2.2)

The differences $\vec{Dh}$, $\vec{De}$ and DE have been calculated in § I,2.4.3 [Equations (I,2.63-65)] /201

$$\vec{Dh} = \vec{h}_{\mathcal{P}} - \vec{h} = \vec{p}_V \wedge \vec{V} + \vec{r} \wedge \dot{\vec{p}}_V \quad (1)$$

$$\vec{De} = \vec{e}_{\mathcal{P}} - \vec{e} = \dot{\vec{p}}_V \wedge \vec{h} + \vec{V} \wedge \vec{Dh} - \frac{\vec{p}_V}{r} + \frac{\vec{r}}{r^3}\left(\vec{r}\cdot\vec{p}_V\right) \quad (2)$$

$$DE = \frac{Da}{2} = \vec{V}\cdot\dot{\vec{p}}_V + \frac{\vec{r}\cdot\vec{p}_V}{r^3} \quad (3)$$

with [see § I,3.2.1., equation (I,3.41)]:

$$\vec{p}_V = 2p_a \vec{V} + \vec{h} \wedge \vec{p}_e + \left(\vec{p}_e \wedge \vec{V} + \frac{\vec{p}_Z}{h}\right) \wedge \vec{r} + p_\tau \left(3M\vec{V} - 2\vec{r}\right). \quad (4)$$

The value of $\dot{\vec{p}}_V = -\vec{p}_r$ calculated on the nominal orbit $(\overline{O})$ (i.e. with constant elements and adjoint) is:

$$\dot{\vec{p}}_V = 2p_a\left(-\frac{\vec{r}}{r^3}\right) + \left[\vec{p}_e \wedge \left(-\frac{\vec{r}}{r^3}\right)\right] \wedge \vec{r} + \left(\vec{p}_e \wedge \vec{V} + \frac{\vec{p}_Z}{h}\right) \wedge \vec{V} + p_\tau\left[3\vec{V} + 3M\left(-\frac{\vec{r}}{r^3}\right) - 2\vec{V}\right] \quad (5)$$

Whence, by introducing $\vec{p}_V$ and $\dot{\vec{p}}_V$ into (1) - (3):

$$\vec{Dh} = \vec{p}_Z \wedge \vec{Z} + \vec{p}_e \wedge \vec{e} - p_\tau \vec{h} \quad (6)$$

$$\vec{De} = \frac{\vec{p}_Z}{\sqrt{1-e^2}} \wedge \vec{e} + \sqrt{1-e^2}\, \vec{p}_e \wedge \vec{Z} \quad (7)$$

$$Da = -2p_\tau \quad (8)$$

equations which coincide with the equations (I,3.47 - 49).

APPENDIX 6

ELEMENTS OF THE MATRIX $B = rKK^{T}$
(see § I,3.2.5)

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	$e \neq 0$	$e = 0$
$B_{\xi\xi}$	$(1-e\cos u)\sin^2 u = 1-e\cos u-\cos^2 u+e\cos^3 u = P_{3\xi\xi}(\cos u)$	$\sin^2 L$
$B_{\eta\eta}$	$\frac{1}{b^2}(\cos u-e)^2(1-e\cos u) =$ $\frac{1}{b^2}\left[e^2-e(2+e^2)\cos u+(1+2e^2)\cos^2 u-e\cos^3 u\right] = P_{3\eta\eta}(\cos u)$	$\cos^2 L$
$B_{\xi\eta}$	$\frac{1}{b}\sin u(\cos u-e)(1-e\cos u) = \sin u\ Q_{2\xi\eta}(\cos u)$	$\sin L\cos L$
B_{aa}	$4(1+e\cos u) = P_{1aa}(\cos u)$	4
$B_{a\alpha}$	$4b^2\cos u = P_{1a\alpha}(\cos u)$	$4\cos L$
$B_{a\beta}$	$4b\sin u = \sin u\ Q_{0a\beta}(\cos u)$	$4\sin L$
$B_{a\tau}$	$-4e\sin u(1-e\cos u)+6M(1+e\cos u)$	$6L$
$B_{\alpha\alpha}$	$b^2(1-3e\cos u+3\cos^2 u-e\cos^3 u) = P_{3\alpha\alpha}(\cos u)$	$1+3\cos^2 L$
$B_{\alpha\beta}$	$b\sin u\left[-e+(3-e^2)\cos u-e\cos^2 u\right] = \sin u\ Q_{2\alpha\beta}(\cos u)$	$3\sin L\cos L$
$B_{\alpha\tau}$	$-2(1-e^2)\sin u(1-e\cos u)+6M(1-e^2)\cos u$	$-2\sin L+6L\cos L$
$B_{\beta\beta}$	$4-3e^2-e(2-e^2)\cos u+(2e^2-3)\cos^2 u+e\cos^3 u = P_{3\beta\beta}(\cos u)$	$4-3\cos^2 L$
$B_{\beta\tau}$	$2\sqrt{1-e^2}(\cos u-e)/(1-e\cos u)+6M\sqrt{1-e^2}\sin u$	$2\cos L+6L\sin L$
$B_{\tau\tau}$	$4(1-e\cos u)^3-12Me\sin u(1-e\cos u)+9M^2(1+e\cos u)$	$4+9L^2$

APPENDIX 7

(induction) (see § I,4.2.2)

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Elements with imposed variation	$e \neq 0$								function of $\vec{p}_v^{\,2}$	$e = 0$						classes making a dbl. use
	d° P	d° Q	$n_{1\,max} \leqslant$	$\Delta\xi$	$\Delta\eta$	Δa	$\Delta\alpha$	$\Delta\beta$		$n_{1\,max} \leqslant$	$\Delta\xi$	$\Delta\eta$	Δa	$\Delta\alpha$	$\Delta\beta$	
ξ	3 (r)		2		0	0	0	0	sin L or cos L	2		0	0	0	0	×
η	3 (r)		2	0		0	0	0	sin L or cos L	2	0		0	0	0	
a	1		1	0	0			0	sin L or cos L	1	0	0		0	0	
α	3		3	0	0			0	sin L or cos L	2	0	0	0		0	
β	3		3	0	0	0	0		sin L or cos L	2	0	0	0	0		×
$\xi\eta$	3 (r)	2 (r)	2	0	0			0	sin L & cos L	2			0	0	0	
ξa	3		3		0			0	sin L or cos L	2		0		0	0	×
$\xi\alpha$	3		3		0			0	sin L or cos L	2		0	0		0	×
$\xi\beta$	3		3		0	0	0		sin L or cos L	2		0	0	0		×
ηa	3		3	0				0	sin L or cos L	2	0			0	0	
$\eta\alpha$	3		3	0				0	sin L or cos L	2	0		0		0	
$\eta\beta$	3		3	0		0	0		sin L or cos L	2	0		0	0		
$a\alpha$	3		3	0	0			0	cos L	2	0	0			0	
$a\beta$	3	0	3	0	0				sin L	2	0	0		0		×
$\alpha\beta$	3	2	3	0	0				sin L & cos L	2	0	0	? → 0			
$\xi\eta a$	3	2	3						sin L & cos L	2				? → 0	? → 0	
$\xi\eta\alpha$	3	2	3						sin L & cos L	2						
$\xi\eta\beta$	3	2	3						sin L & cos L	2						×
$\xi a\alpha$	3		3		0			0	cos L	2		0			0	×
$\xi a\beta$	3	0	3						sin L	2		0		0		×
$\xi\alpha\beta$	3	2	3						sin L & cos L	2						×
$\eta a\alpha$	3		3	0				0	cos L	2	0				0	
$\eta a\beta$	3	0	3						sin L	2	0			0		
$\eta\alpha\beta$	3	2	3						sin L & cos L	2						
$a\alpha\beta$	3	2	3	0	0				sin L & cos L	2	0	0				
$\xi\eta a\alpha$	3	2	3						sin L & cos L	2						
$\xi\eta a\beta$	3	2	3						sin L & cos L	2						×
$\xi\eta\alpha\beta$	3	2	3						sin L & cos L	2						
$\xi a\alpha\beta$	3	2	3						sin L & cos L	2						×
$\eta a\alpha\beta$	3	2	3						sin L & cos L	2						
$\xi\eta a\alpha\beta$	3	2	3						sin L & cos L	2						

APPENDIX -8

ELEMENTS OF THE MATRIX $G = \int_{u_0}^{u_f} B\,du$ (System S_2)
(see § I,3.3)

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	$e \neq 0$	$e = 0$
$G_{\xi\xi}$	$\left[\frac{u}{2} - \frac{\sin u \cos u}{2} - \frac{e}{3}\sin^3 u\right]_{u_0}^{u_f}$	$\frac{1}{2}\left[L - \sin L \cos L\right]_{L_0}^{L_f}$
$G_{\xi\eta}$	$-\frac{1}{\sqrt{1-e^2}}\left[e\cos u - \frac{1+e^2}{2}\cos^2 u + \frac{e}{3}\cos^3 u\right]_{u_0}^{u_f}$	$\frac{1}{2}\left[\cos^2 L\right]_{L_0}^{L_f}$
$G_{\eta\eta}$	$\frac{1}{1-e^2}\left[\frac{1+4e^2}{2}u - e(3+e^2)\sin u + \frac{1+e^2}{2}\sin u \cos u + \frac{e}{3}\sin^3 u\right]_{u_0}^{u_f}$	$\frac{1}{2}\left[L + \sin L \cos L\right]_{L_0}^{L_f}$
G_{aa}	$4\left[u + e\sin u\right]_{u_0}^{u_f}$	$4\left[L\right]_{L_0}^{L_f}$
$G_{a\alpha}$	$4(1-e^2)\left[\sin u\right]_{u_0}^{u_f}$	$4\left[\sin L\right]_{L_0}^{L_f}$
$G_{a\beta}$	$-4\sqrt{1-e^2}\left[\cos u\right]_{u_0}^{u_f}$	$-4\left[\cos L\right]_{L_0}^{L_f}$
$G_{a\tau}$	$\left[3u^2 + 6eu\sin u + 16e\cos u - e^2\sin^2 u\right]_{u_0}^{u_f} \underset{\Delta u \to \infty}{\sim} 3\Delta u^2$	$\left[3L^2\right]_{L_0}^{L_f}$
$G_{\alpha\alpha}$	$(1-e^2)\left[\frac{5}{2}u - 4e\sin u + \frac{3}{2}\sin u\cos u + \frac{e}{3}\sin^3 u\right]_{u_0}^{u_f}$	$\frac{1}{2}\left[5L + 3\sin L \cos L\right]_{L_0}^{L_f}$
$G_{\alpha\beta}$	$\sqrt{1-e^2}\left[e\cos u - \frac{3-e^2}{2}\cos^2 u + \frac{e}{3}\cos^3 u\right]_{u_0}^{u_f}$	$-\frac{3}{2}\left[\cos^2 L\right]_{L_0}^{L_f}$
$G_{\alpha\tau}$	$2(1-e^2)\left[3u\sin u + 4\cos u - e\sin^2 u\right]_{u_0}^{u_f} \underset{\Delta u \to \infty}{\sim}$ order Δu	$\left[6L\sin L + 8\cos L\right]_{L_0}^{L_f}$
$G_{\beta\beta}$	$\left[\frac{5-4e^2}{2}u - e(1-e^2)\sin u - \frac{3-2e^2}{2}\sin u\cos u - \frac{e}{3}\sin^3 u\right]_{u_0}^{u_f}$	$\frac{1}{2}\left[5L - 3\sin L \cos L\right]_{L_0}^{L_f}$
$G_{\beta\tau}$	$2\sqrt{1-e^2}\left[-3eu - 3u\cos u + (4+e^2)\sin u + e\sin u\cos u\right]_{u_0}^{u_f} \underset{\Delta u \to \infty}{\sim}$ order Δu	$\left[-6L\cos L + 8\sin L\right]_{L_0}^{L_f}$
$G_{\tau\tau}$	$\left[3u^3 + 9eu^2\sin u + u(4 + 48e\cos u + 18e^2 - 3e^2\sin^2 u) - 60e\sin u - 6e^2\sin u\cos u + e^3\left(\frac{\sin^2 u}{3} - 4\right)\sin u\right]_{u_0}^{u_f} \underset{\Delta u \to \infty}{\sim} 3\Delta u^2$	$\left[3L^3 + 4L\right]_{L_0}^{L_f}$

APPENDIX 9

DIAGONAL OF THE MATRIX G [system (S_2)]

(see § I,4.1.2)

It is enough to consider the plane case. Then let us posit: /209

$$\Delta \mathcal{X} = \begin{vmatrix} \Delta a \\ \Delta \alpha \\ \Delta \beta \\ \Delta \tau \end{vmatrix} \tag{1}$$

We propose to take $\Delta\tau_1$ as the fourth variation instead of $\Delta\tau$ so as to diagonalize the matrix G. If:

$$\Delta \mathcal{X}_1 = \begin{vmatrix} \Delta a \\ \Delta \alpha \\ \Delta \beta \\ \Delta \tau_1 \end{vmatrix} \tag{2}$$

designates the new variation of the kinematic state, this is the same as carrying out the linear transformation:

$$\Delta \mathcal{X}_1 = A\,\Delta \mathcal{X} \tag{3}$$

where A is a matrix of the form:

$$A = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \ell & m & n & k \end{vmatrix} \tag{4}$$

Since consumption ΔJ is an invariant, A must be such that:

$$\Delta \mathcal{X}_1^T G_1^{-1} \Delta \mathcal{X}_1 = \Delta \mathcal{X}^T G^{-1} \Delta \mathcal{X} \tag{5}$$

no matter what ΔX may be. From this identity we derive:

$$G_1 = AGA^T = \left[\begin{array}{c|c|c|c} G_{aa} & 0 & 0 & \ell G_{aa} + kG_{a\tau} \\ 0 & G_{\alpha\alpha} & 0 & mG_{\alpha\alpha} + kG_{\alpha\tau} \\ 0 & 0 & G_{\beta\beta} & nG_{\beta\beta} + kG_{\beta\tau} \\ \ell G_{aa} + kG_{a\tau} & mG_{\alpha\alpha} + kG_{\alpha\tau} & nG_{\beta\beta} + kG_{\beta\tau} & \begin{array}{l} \ell(\ell G_{aa} + kG_{a\tau}) \\ + m(mG_{\alpha\alpha} + kG_{\alpha\tau}) \\ + n(nG_{\beta\beta} + kG_{\beta\tau}) \\ + k(\ell G_{a\tau} + mG_{\alpha\tau} \\ \quad + nG_{\beta\tau} + kG_{\tau\tau}) \end{array} \end{array}\right] \tag{6}$$

This matrix can be diagonalized by choosing A so that:

$$\ell = -k\frac{G_{a\tau}}{G_{aa}} \tag{7}$$ /210

$$m = -k\frac{G_{\alpha\tau}}{G_{\alpha\alpha}} \tag{8}$$

$$n = -k\frac{G_{\beta\tau}}{G_{\beta\beta}} \; . \tag{9}$$

Then:

$$G_{1\tau\tau} = k^2\left(G_{\tau\tau} - \frac{G_{a\tau}^2}{G_{aa}} - \frac{G_{\alpha\tau}^2}{G_{\alpha\alpha}} - \frac{G_{\beta\tau}^2}{G_{\beta\beta}}\right). \tag{10}$$

Therefore a degree of freedom still exists in the choice of A (parameter k). The change of the variable is written:

$$\Delta\tau_1 = k\left(\Delta\tau - \Delta\tau_{t.s}\right) \tag{11}$$

where:

$$\Delta\tau_{t.s} = \frac{G_{a\tau}}{G_{aa}}\,\Delta a + \frac{G_{\alpha\tau}}{G_{\alpha\alpha}}\,\Delta\alpha + \frac{G_{\beta\tau}}{G_{\beta\beta}}\,\Delta\beta \tag{12}$$

is nothing but the variation of τ *in the simple transfer* (achieved in the optimal way) corresponding to the rendezvous under consideration, *induced* by the variatons of a, α and β. In the case of simple transfer let us actually posit:

$$\Delta \mathcal{O} = \begin{bmatrix} \Delta a \\ \Delta\alpha \\ \Delta\beta \\ 0 \end{bmatrix} \quad \text{(fixed since } \Delta a,\ \Delta\alpha,\ \Delta\beta \text{ are given)} \tag{13}$$

The adjoint referring to the simple transfer is, on the other hand:

$$P_{t.s.} = \left[P_{a,t.s.},\ P_{\alpha,t.s.},\ P_{\beta,t.s.},\ 0 \right] \tag{14}$$

Then equation (I,3 - 66) is written:

$$\Delta \mathcal{X}_{t.s.} = \Delta \mathcal{O} + Q\,\Delta\tau_{t.s.} = G P^T_{t.s.} \tag{15}$$

where Q is the matrix:

$$Q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{16}$$

In simple transfer, $\Delta\tau_{t.s}$ and $P_{t.s}$ must be considered as unknowns to be /211
calculated as a function of the transfer ΔO.

These quantities are given by the matrix equation:

$$G P^T_{t.s.} - Q\,\Delta\tau_{t.s.} = S\left[P^T_{t.s.} + Q\,\Delta\tau_{t.s.} \right] = \Delta\mathcal{O} \tag{17}$$

where S is the matrix:

$$S = \begin{bmatrix} G_{aa} & 0 & 0 & 0 \\ 0 & G_{\alpha\alpha} & 0 & 0 \\ 0 & 0 & G_{\beta\beta} & 0 \\ G_{a\tau} & G_{\alpha\tau} & G_{\beta\tau} & -1 \end{bmatrix} \tag{18}$$

Whence:

$$\Delta\tau_{t.s.} = Q^T S^{-1} \Delta\mathcal{O} \tag{19}$$

with

$$S^{-1} = \begin{array}{|c|c|c|c|} \hline \frac{1}{G_{aa}} & 0 & 0 & 0 \\ \hline 0 & \frac{1}{G_{\alpha\alpha}} & 0 & 0 \\ \hline 0 & 0 & \frac{1}{G_{\beta\beta}} & 0 \\ \hline \frac{G_{a\tau}}{G_{aa}} & \frac{G_{\alpha\tau}}{G_{\alpha\alpha}} & \frac{G_{\beta\tau}}{G_{\beta\beta}} & -1 \\ \hline \end{array} \tag{20}$$

$\Delta\tau_{t.s}$ is given very well by expression (12).

If k is taken as equal to unity, $\Delta\tau_1$ then represents the necessary *supplement* to assure rendezvous.

The supplementary term ΔJ_1 being introduced into ΔJ is equal to:

$$\Delta J_1 = \frac{\Delta\tau_1^2}{2G_{1\tau\tau}} . \tag{21}$$

In the case of large number of turns ($N >> 1$), $G_{\tau\tau} \sim 3\Delta u^3$, $G_{a\tau} \sim -3\Delta u^2$, $G_{\alpha\tau}$ and $G_{\beta\tau}$ = order Δu, therefore $G_1 \sim \frac{3}{4}\Delta u^3$ and:

$$\Delta J_1 = \frac{2\Delta\tau_1^2}{3\Delta u^3} . \tag{22}$$

APPENDIX 10

INFINITESIMAL ROTATION OF THE PLANE OF THE ORBIT

SYSTEM S_{1c} ($W = C^{te}$, $F \leq F_{max}$, CONTINUOUS THRUST)

(see § II,2.3.2.2)

$e = 0 \quad \alpha = \delta \quad |\lambda| = |\nu| = \frac{2}{\pi} = 0,637$ /213

$e = 0,1$

$\alpha°$	$\delta°$	$\|\lambda\| = \|\nu\|$
0	0	0.650
10	9° 40'	0.649
20	19° 22'	0.648
30	29° 09'	0.646
40	39° 01'	0.645
50	19° 01'	0.642
60	59° 08'	0.640
70	69° 21'	0.639
80	79° 39'	0.637
90	90°	0.637

$e = 0,2$

$\alpha°$	$\delta°$	$\|\lambda\| = \|\nu\|$
0	0	0.690
10	8° 43'	0.688
20	17° 35'	0.682
30	26° 42'	0.676
40	36° 11'	0.669
50	46° 07'	0.660
60	56° 32'	0.651
70	67° 24'	0.643
80	78° 36'	0.640
90	90°	0.637

$e = 0,3$

$\alpha°$	$\delta°$	$\|\lambda\| = \|\nu\|$
0	0	0.756
10	7° 24'	0.755
20	14° 56'	0.746
30	23° 01'	0.735
40	31° 49'	0.716
50	41° 31'	0.695
60	52° 16'	0.674
70	64° 05'	0.655
80	76° 47'	0.642
90	90°	0.637

$e = 0,4$

0	0	0.860
10	5° 45'	0.855
20	11° 51'	0.845
30	18° 38'	0.825
40	26° 25'	0.794
50	35° 36'	0.756
60	46° 29'	0.715
70	59° 21'	0.676
80	74° 04'	0.648
90	90°	0.637

$e = 0,5$

0	0	1.005
10	4° 10'	1.00
20	8° 45'	0.985
30	14° 05'	0.959
40	20° 39'	0.915
50	28° 55'	0.856
60	39° 32'	0.786
70	53° 12'	0.715
80	70° 18'	0.660
90	90°	0.637

$e = 0,6$

0	0	1.212
10	2° 45'	1.205
20	5° 55'	1.19
30	9° 53'	1.155
40	15° 07'	1.10
50	22° 10'	1.017
60	31° 55'	0.905
70	45° 43'	0.784
80	65° 12'	0.680
90	90°	0.637

$e = 0,7$

0	0	1.520
10	1° 35'	1.515
20	3° 34'	1.498
30	6° 18'	1.460
40	10° 14'	1.389
50	15° 53'	1.270
60	24° 13'	1.105
70	37° 08'	0.908
80	58° 19'	0.721
90	90°	0.637

$e = 0,8$

0	0	2.02
10	0° 44'	2.02
20	1° 47'	2.005
30	3° 30'	1.96
40	6° 12'	1.86
50	10° 23'	1.701
60	16° 53'	1.461
70	27° 46'	1.142
80	48° 53'	0.810
90	90°	0.637

$e = 0,9$

0	0	3.10
10	0° 12'	3.10
20	0° 37'	3.09
30	1° 28'	3.04
40	3° 24'	2.91
50	5° 41'	2.68
60	9° 58'	2.26
70	17° 35'	1.701
80	35° 08'	1.05
90	90°	0.637

APPENDIX 11

OPTIMAL ROTATION OF THE PLANE OF AN ORBIT WITH WEAK ECCENTRICITY (see § II,2.3.2.2)

1. ZONE 2 - TWO THRUST ZONES.

$$0 \leqslant s \leqslant 1 - e \cos \alpha. \tag{1}$$ /215

Taking e as an infinitely small principal and noticing that s $\lessgtr$ 1 (unless $\alpha = \frac{\pi}{2}$), it becomes:

$$\sin \beta' = (1 - \cos^2 \beta')^{1/2} = \left[1 - (e \cos \alpha + s)^2\right]^{1/2} =$$
$$\sqrt{1 - s^2} - \frac{e\, s \cos \alpha}{\sqrt{1 - s^2}} - \frac{e^2 \cos^2 \alpha}{2(1 - s^2)\sqrt{1 - s^2}} + \text{order } e^3 \tag{2}$$

$$\sin \beta'' = (1 - \cos^2 \beta'')^{1/2} = \left[1 - (e \cos \alpha - s)^2\right]^{1/2} = \tag{3}$$
$$\sqrt{1 - s^2} + \frac{e\, s \cos \alpha}{\sqrt{1 - s^2}} - \frac{e^2 \cos^2 \alpha}{2(1 - s^2)\sqrt{1 - s^2}} + \text{order } e^3$$

$$\sin \beta' + \sin \beta'' = 2\sqrt{1 - s^2} - \frac{e^2 \cos^2 \alpha}{2(1 - s^2)\sqrt{1 - s^2}} + \text{order } e^4 \tag{4}$$

$$\sin \beta' - \sin \beta'' = -\frac{2\, e\, s \cos \alpha}{\sqrt{1 - s^2}} + \text{order } e^3 \tag{5}$$

and, by Taylor's formula:

$$\beta' = \text{Arc} \sin\left(\sqrt{1 - s^2} - \frac{e\, s \cos \alpha}{\sqrt{1 - s^2}} - \frac{e^2 \cos^2 \alpha}{2(1 - s^2)\sqrt{1 - s^2}} + \text{order } e^3\right) =$$
$$\text{Arc} \cos s - \frac{e \cos \alpha}{\sqrt{1 - s^2}} - \frac{e^2 s \cos^2 \alpha}{2(1 - s^2)\sqrt{1 - s^2}} + \text{order } e^3 \tag{6}$$

and by changing s into -s: /216

$$\beta'' = \pi - \operatorname{Arc}\cos s - \frac{e\cos\alpha}{\sqrt{1-s^2}} + \frac{e^2 s\cos^2\alpha}{2(1-s^2)\sqrt{1-s^2}} + \text{order}\, e^3 \tag{7}$$

whence:

$$\beta' + \beta'' - \pi = -\frac{2e\cos\alpha}{\sqrt{1-s^2}} + \text{order}\, e^3 \tag{8}$$

$$\beta' + \pi - \beta'' = 2\operatorname{Arc}\cos s - \frac{e^2 s\cos^2\alpha}{(1-s^2)\sqrt{1-s^2}} + \text{order}\, e^4. \tag{9}$$

On the other hand:

$$\sin 2\beta' + \sin 2\beta'' = 4e\cos\alpha\frac{1-2s^2}{\sqrt{1-s^2}} + \text{order}\, e^3. \tag{10}$$

Introducing all these values into (II,2. 24-26), it becomes:

$$\lambda_\eta = \frac{\Delta\eta}{2N\pi F_{max}} = \frac{-1}{2\pi\sqrt{1-e^2}}\left[4\sqrt{1-s^2}\cos\alpha + \frac{2e^2}{\sqrt{1-s^2}}\cdot \cos\alpha\left(6-4s^2-\cos^2\alpha\,\frac{3-6s^2+4s^4}{1-s^2}\right) + \text{order}\, e^3\right] \tag{11}$$

$$\lambda_\xi = \frac{\Delta\xi}{2N\pi F_{max}} = \frac{1}{2\pi}\left[4\sqrt{1-s^2}\sin\alpha - \frac{2e^2}{\sqrt{1-s^2}}\sin\alpha\cos^2\alpha\,\frac{3-6s^2+4s^4}{1-s^2} + \text{order}\, e^3\right] \tag{12}$$

$$\lambda_c = \frac{W|\Delta m|}{2N\pi F_{max}} = \frac{1}{2\pi}\left[4\operatorname{Arc}\cos s + \frac{2s(1-2s^2)}{(1-s^2)\sqrt{1-s^2}}e^2\cos^2\alpha + \text{order}\, e^3\right] \tag{13}$$

$$\lambda = \frac{|\Delta j|}{2N\pi F_{max}} = \frac{1}{2\pi}\left[4\sqrt{1-s^2} + 2\frac{4-6s^2+s^4}{(1-s^2)\sqrt{1-s^2}}e^2\cos^2\alpha + \text{order}\, e^3\right] \tag{14}$$

In (13) and (14), α can be replaced by its value for $e = 0$, that is to say by δ. We can then derive s from (14):

$$s = \sqrt{1-z^2}\left(1 + \frac{z^4+4z^2-1}{2z^2(1-z^2)}\, e^2 \cos^2\delta + \text{order } e^3\right) \tag{15}$$

with /217

$$z = \frac{\pi}{2}\,\lambda \tag{16}$$

and introducing this value into (13) it becomes:

$$\lambda_c = \frac{1}{2\pi}\left[4\ \text{Arc}\sin z - \frac{2(1+3z^2)}{z\sqrt{1-z^2}}\, e^2\cos^2\delta + \text{order } e^3\right] \tag{17}$$

whence the specific rotation:

$$\nu = \frac{|\Delta j|}{W/|\Delta m|} = \frac{z}{\text{Arc}\sin z} + \frac{1+3z^2}{4(\text{Arc}\sin z)^2\sqrt{1-z^2}}\, e^2\cos^2\delta + \text{order } e^4 \tag{18}$$

This formula *directly* gives the specific rotation as a function of the rotation to be achieved, of the number of turns permitted and of the maximal thrust with rather close precision when the eccentricity is weak.

Condition (1) is also written:

$$s \ll 1 - e\cos\delta + \text{order } e^2. \tag{19}$$

We shall demonstrate that then:

$$z \gg \sqrt{e\cos\delta} + \text{order } e^{3/2}. \tag{20}$$

When we approach the frontier (Γ'), z diminishes and becomes on the order of $e^{1/2}$. Formula (18) then shows that the second term of the development is of the order e^2/z^2, i.e. of the order e and not of the order e^2.

The error is then of the order of e^2 and not of e^4.

The result is that, beyond (Γ'), it will be enough to develop as far as the terms in e (error of the order of e^2) without precision being less than that obtained by (18) in the vicinity of (Γ').

2. A SINGLE THRUST ZONE.

So that there may be only one thrust zone, it is both necessary and sufficient for:

$$\begin{cases} 1-e\cos\alpha < s \leqslant 1+e\cos\alpha \\ \alpha \neq \frac{\pi}{2} \end{cases} \tag{21}$$

Then let us posit:

$$s = 1 + k\, e \cos\alpha \tag{22}$$

with:

$$-1 < k \leqslant 1 \tag{23}$$

k replaces the parameter.

Then it becomes:

$$\cos\beta'' = e\cos\alpha - s = -1 + (1-k)\, e\cos\alpha \tag{24}$$

$$\sin\beta'' = \left[2(1-k)\cos\alpha\right]^{1/2} e^{1/2} - \frac{1}{8}\left[2(1-k)\cos\alpha\right]^{3/2} e^{3/2} + \text{order } e^{5/2} \tag{25}$$

$$\pi - \beta'' = \left[2(1-k)\cos\alpha\right]^{1/2} e^{1/2} + \frac{1}{24}\left[2(1-k)\cos\alpha\right]^{3/2} e^{3/2} + \text{order } e^{5/2} \tag{26}$$

$$\sin 2\beta'' = -2\left[2(1-k)\cos\alpha\right]^{1/2} e^{1/2} + \text{order } e^{3/2} \tag{27}$$ /218

$$\lambda_\eta = \frac{\Delta\eta}{2N\pi F_{max}} = \frac{-1}{2\pi\sqrt{1-e^2}}\left\{2\cos\alpha\left[2(1-k)\cos\alpha\right]^{1/2} e^{1/2} + 2\left[2(1-k)\cos\alpha\right]^{1/2}\left(\frac{3+k}{4}\cos^2\alpha + 1\right) e^{3/2} + \text{order } e^{5/2}\right\} \tag{28}$$

$$\lambda_\xi = \frac{\Delta\xi}{2N\pi F_{max}} = \frac{1}{2\pi}\left\{2\sin\alpha\left[2(1-k)\cos\alpha\right]^{1/2} e^{1/2} + 2\left[2(1-k)\cos\alpha\right]^{1/2}\frac{3+k}{4}\sin\alpha\cos\alpha\, e^{3/2} + \text{order } e^{5/2}\right\} \tag{29}$$

$$\lambda = \frac{|\Delta j|}{2N\pi F_{max}} = \frac{1}{2\pi}\left\{2\left[2(1-k)\cos\alpha\right]^{1/2}\right. \tag{30}$$

$$\left.\left(e^{1/2} + \frac{7+k}{4}\cos\alpha\, e^{3/2} + \text{order } e^{5/2}\right)\right\}$$

$$\lambda_c = \frac{W|\Delta m|}{2N\pi F_{max}} = \frac{1}{2\pi}\left\{2\left[2(1-k)\cos\alpha\right]^{1/2} e^{1/2} + \right. \tag{31}$$

$$\left.\frac{13-k}{6}\cos\alpha\left[2(1-k)\cos\alpha\right]^{1/2} e^{3/2} + \text{order } ^{5/2}\right\}$$

$$\nu = \frac{|\Delta j|}{W|\Delta m|} = \frac{\lambda}{\lambda_c} = 1 + \frac{k+2}{3}\, e\cos\alpha + \text{order } e^2 \tag{32}$$

$$\cos\delta = \cos\alpha + e\sin^2\alpha + \text{order } e^2. \tag{33}$$

In the second term of (32), it is enough to introduce the values α and k derived from (30) and (33) where only the first term of development is maintained:

$$\alpha = \delta \tag{34}$$

$$k = 1 - \frac{\pi^2\lambda^2}{2e\cos\delta} \tag{35}$$

whence:

$$\nu = 1 + e\cos\delta - \frac{\pi^2\lambda^2}{6} + \text{order } e^2. \tag{36}$$ /219

This is a formula to be compared with (18).

The points of the frontier (Γ') correspond to k = -1 or, by (35):

$$z = \frac{\pi\lambda}{2} = \sqrt{e\cos\delta} + \text{order } ^{3/2}. \tag{37}$$

Introducing these into (18) and (36), we obtain the same development:

$$\nu(\Gamma') = 1 + \frac{e \cos \delta}{3} + \text{order}\, e^2. \tag{38}$$

In the particular case of the circle, expression of the specific rotation is very simple:

$$\nu = \frac{|\Delta j|}{W/|\Delta m|} = \frac{\frac{\pi}{2}\lambda}{\operatorname{Arc\,sin}\left(\frac{\pi}{2}\lambda\right)} \quad \text{where } \lambda = \frac{|\Delta j|}{2N\pi F_{max}}. \tag{39}$$

APPENDIX 12

TRANSFER BETWEEN CIRCLES - SYSTEM (S_1)
(see § II,3.3.2)

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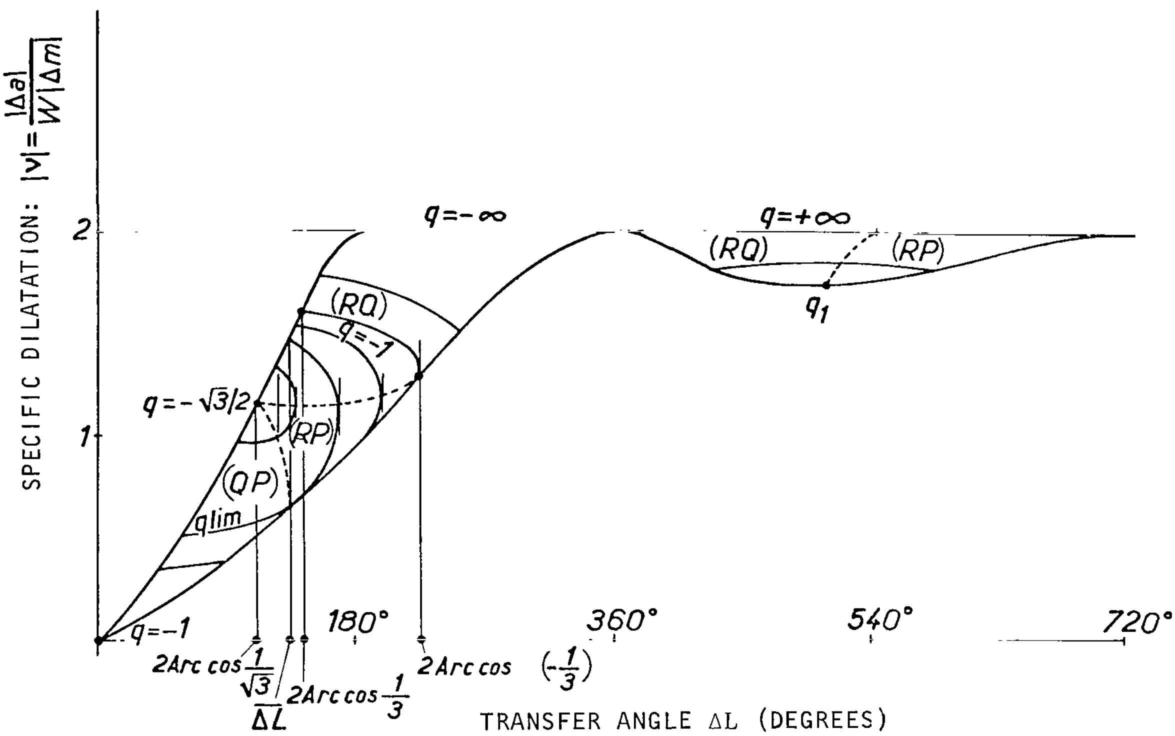

Fig. 1. Iso-q (qualitative) Lines (System S_1).

1. $\boxed{0 \leqslant \Delta L \leqslant 2\pi}$

SOLUTIONS CORRESPONDING TO A PARTICULAR VALUE OF q(Figure 1).

For a value of $-\sqrt{3}/2 \leq q \leq 0$, there is no solution for (II,3 -41) cannot be satisfied.

For $q = -\sqrt{3}/2$ (Figure 2a), there is only one bi-impulsional solution corresponding to the point $x_o = x_1 = 1/\sqrt{3}$, i.e. to the transfer angle $\Delta L = 2 \text{ Arc cos } \frac{1}{\sqrt{3}} = 109° \ 28'$.

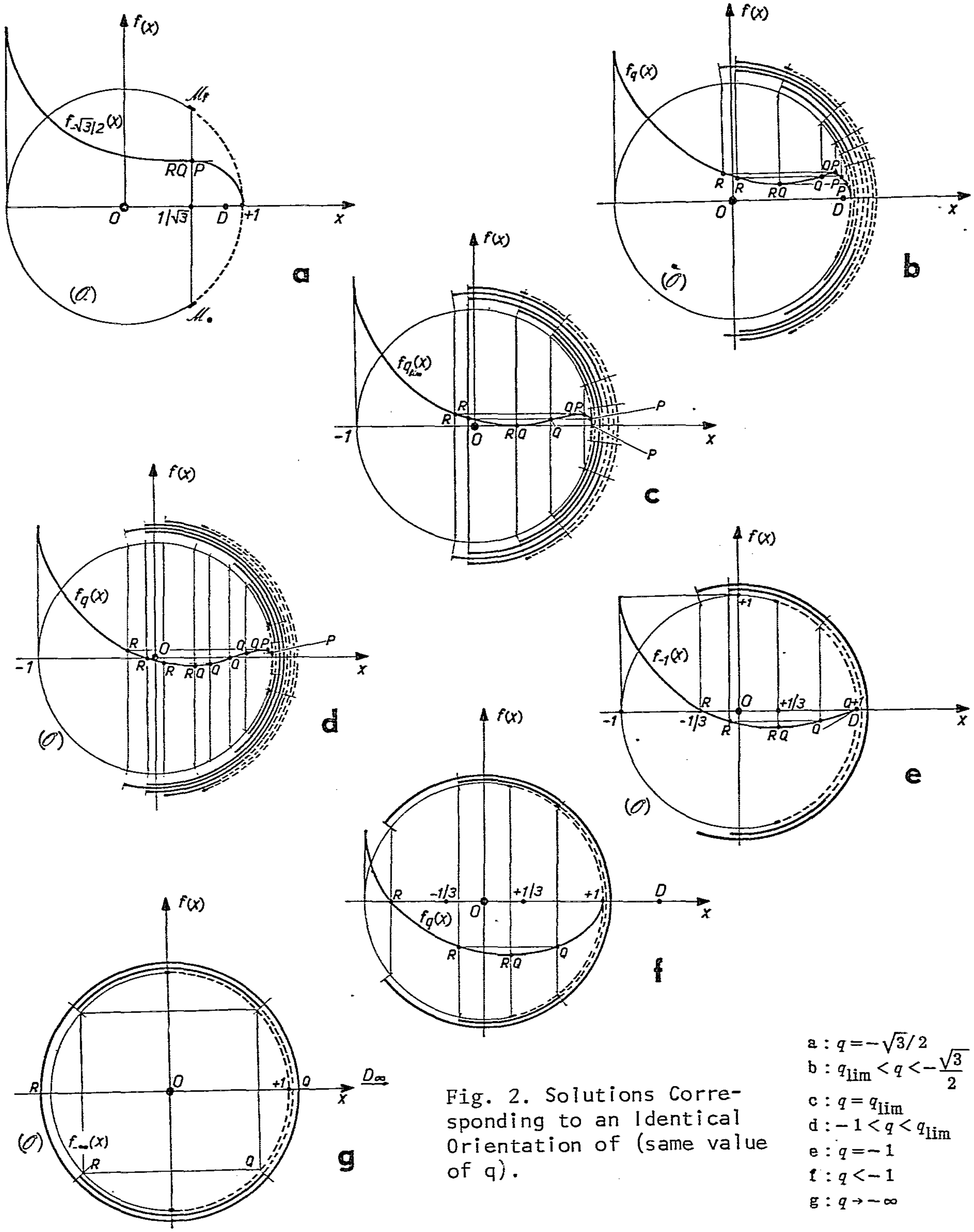

Fig. 2. Solutions Corresponding to an Identical Orientation of (same value of q).

a : $q = -\sqrt{3}/2$
b : $q_{\text{lim}} < q < -\frac{\sqrt{3}}{2}$
c : $q = q_{\text{lim}}$
d : $-1 < q < q_{\text{lim}}$
e : $q = -1$
f : $q < -1$
g : $q \rightarrow -\infty$

For $q_{1im} < q < -\sqrt{3}/2$ (Figure 2b), there is an infinity of solutions corresponding *to different values of the transfer angle and of the ratio* $|\lambda_1| = |\Delta a|/F_{max}$. These solutions can be followed by continuity when the transfer angle varies, from the bi-impulsional solution (QP) by going through an entire gamut of solutions with two thrust arcs. It will be noticed that /223
among these solutions there is no solution where thrust is applied constantly.

For $q = q_{1im}$ (Figure 2c), there appears one solution where thrust is applied constantly which we shall call the "continuous thrust" solution.

For $-1 < q < q_{1im}$ (Figure 2d), there are two "continuous thrust" solutions.

For $q = -1$ (Figure 2e), the bi-impulsional solution is unique and corresponds to $x = 1/3$ or

$$\Delta L = 2 \text{ Arc cos } \frac{1}{3} = 141°04'.$$

The "continuous thrust" solution is unique and corresponds to $x_o = -1/3$ or

$$\Delta L = 2 \text{ Arc cos } (-1/3) = 218°56'.$$

For $q < -1$ (Figure 2f), the bi-impulsional solution and the "continuous thrust" solution are unique.

When $q \to -\infty$ (Figure 2g), we obtain, *by continuity*, a bi-impulsional Hohmann solution ($\Delta L = 180°$), solutions of the "Hohmann type" including two *symmetrical* thrust arcs with reference to 0, and finally a "continuous thrust" solution corresponding to one revolution ($\Delta L = 360°$).

We have seen that actually, for $p_e = 0$, $p_a = -1/2$ and therefore $q = -\infty$, the solution is singular and cannot be obtained by the preceding method. The solutions obtained above form part of a larger family of solutions.

SOLUTIONS CORRESPONDING TO A PARTICULAR VALUE OF ΔL.

The preceding results can be presented in a different manner, more conformable to the practice where the transfer angle is fixed.

Let there be a transfer angle to be achieved with transfer ΔL given ($0 \leq \Delta L \leq 2\pi$). The abscissa x_o of Figure II,3 - 12 is fixed.

In general there is an infinity of solutions, *corresponding to the different values of* q *(therefore of* x_1*) and of the relationship* $|\lambda_1| = |\Delta a|/F_{max}$. By ΔL let us designate the transfer angle corresponding to an intersection of the locus (L) of the maxima and minima of $f(x)$ with $\overrightarrow{Ox}$ (Figure II,3 - 11).

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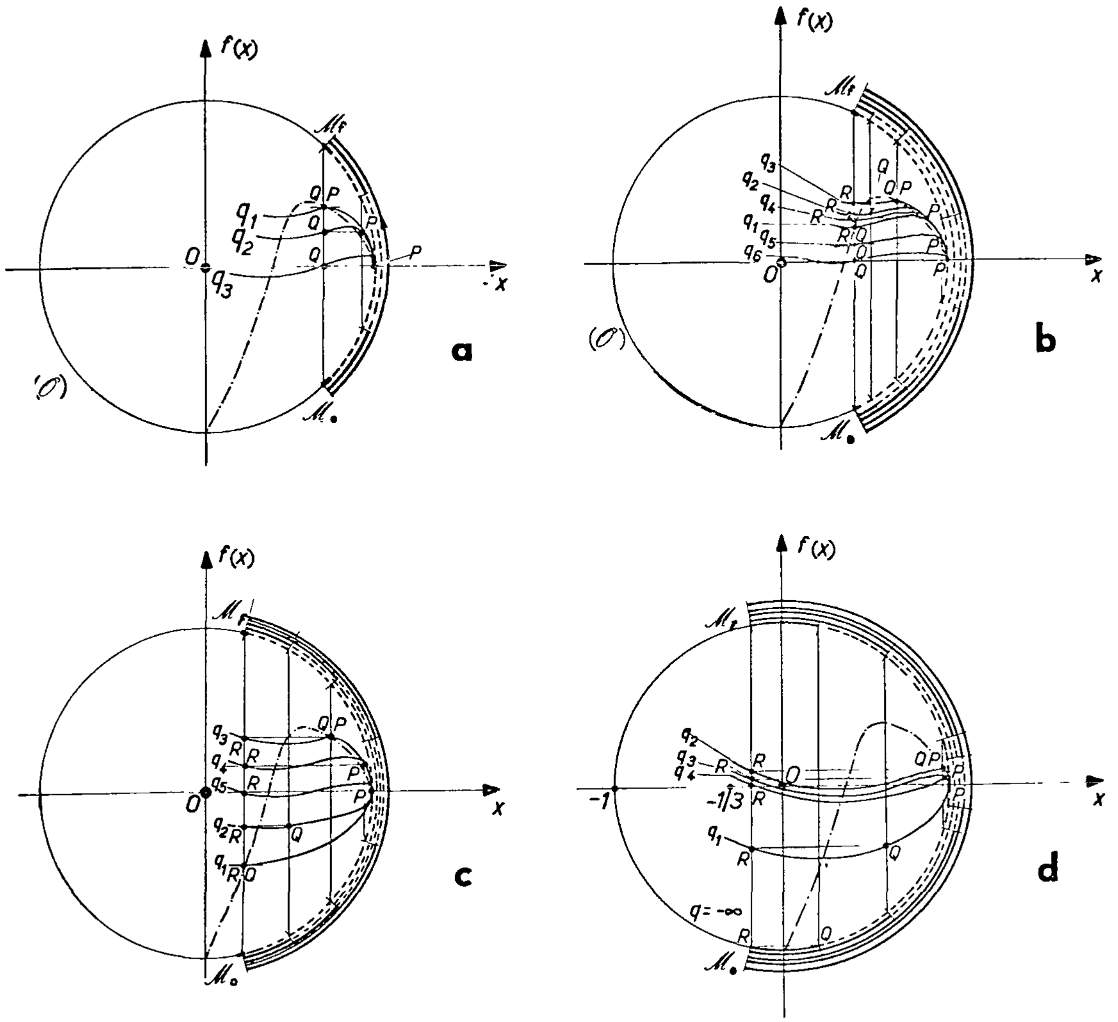

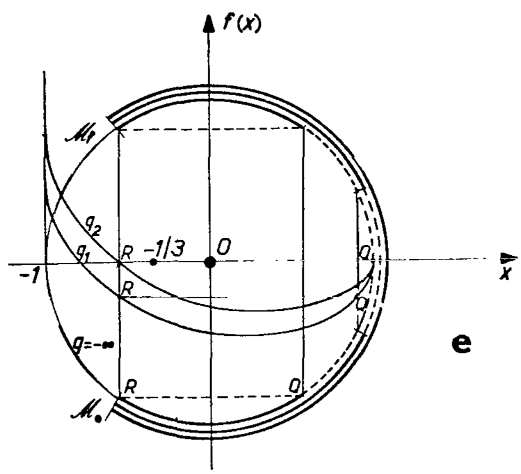

Fig. 3. Solutions Corresponding to the Same Transfer Angle ΔL.

a : $0 \leqslant \Delta L \leqslant 2 \text{ Arc cos } (1/\sqrt{3})$
b : $2 \text{ Arc cos } 1/\sqrt{3} < \Delta L < \overline{\Delta L}$
c : $\overline{\Delta L} < \Delta L < \pi$
d : $\pi < \Delta L < 2 \text{ Arc cos } (-1/3)$
e : $2 \text{ Arc cos } (-1/3) < \Delta L < 2\pi$

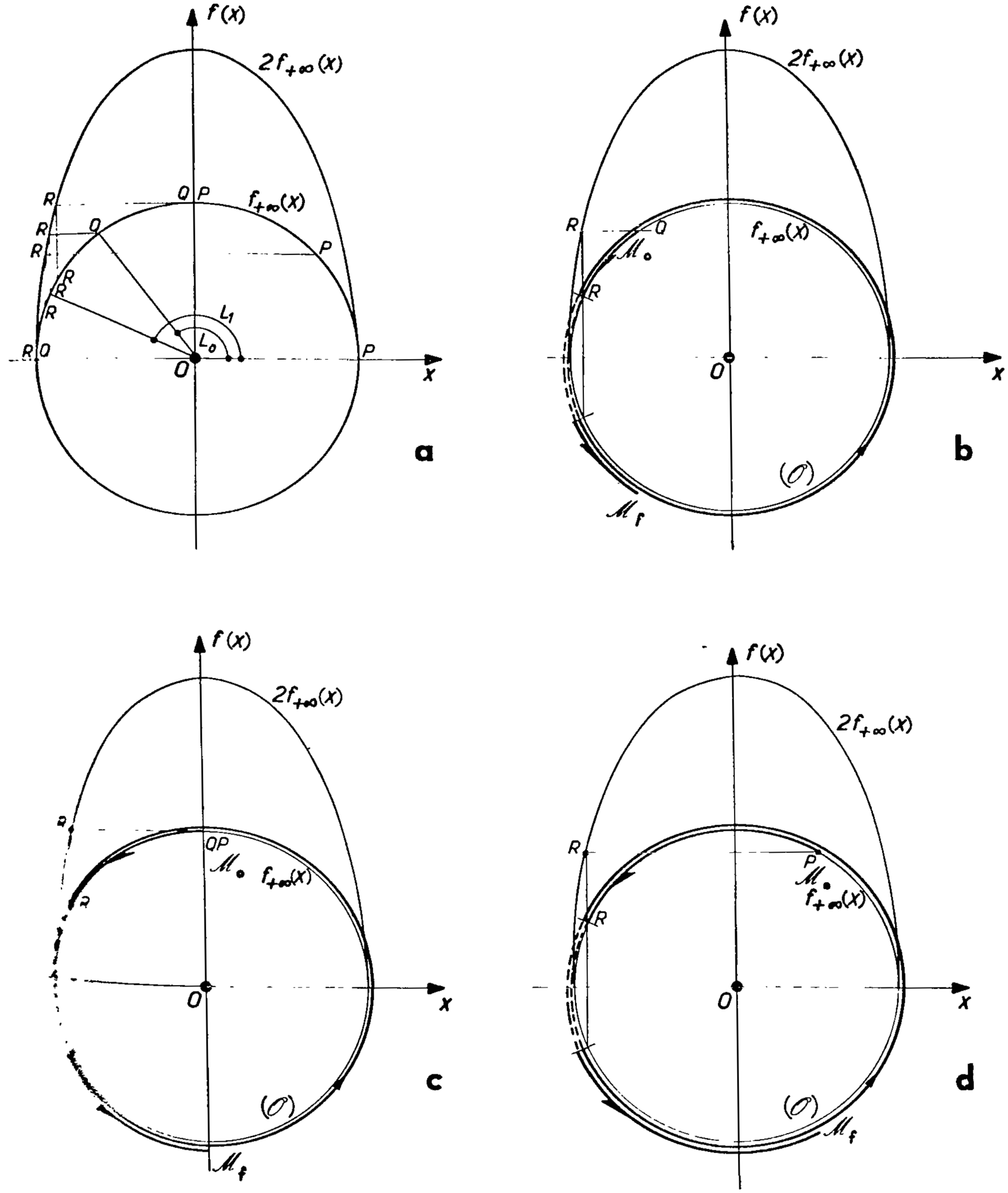

Fig. 4. $2\pi \leq \Delta L \leq 4\pi$. Solutions Corresponding to a Tangential Thrust ($q \to +\infty$)
a: Type of Solutions; b: RQ Solutions; c: R(QP) Solutions; d: RP Solutions.

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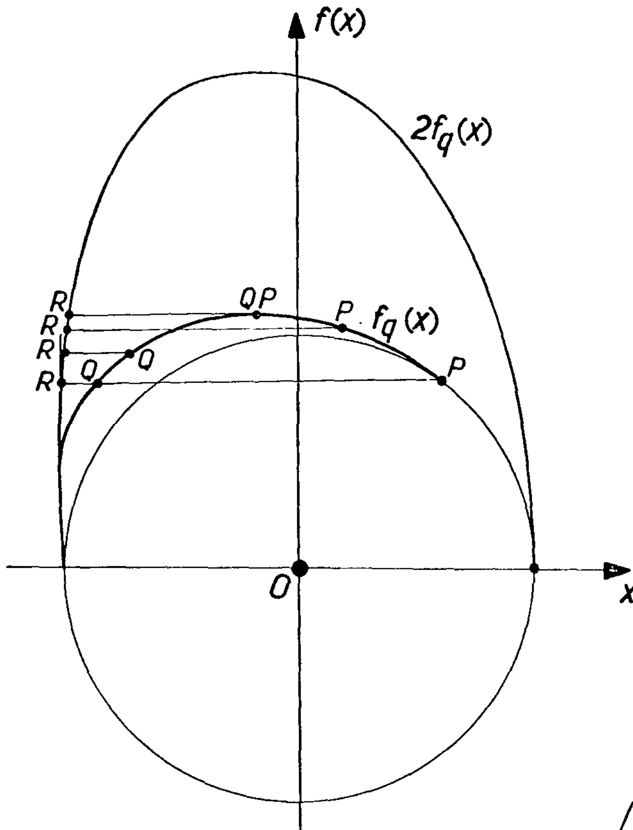

Fig. 5 - $2\pi \leqslant \Delta L \leqslant 4\pi$. $(q > q_1)$

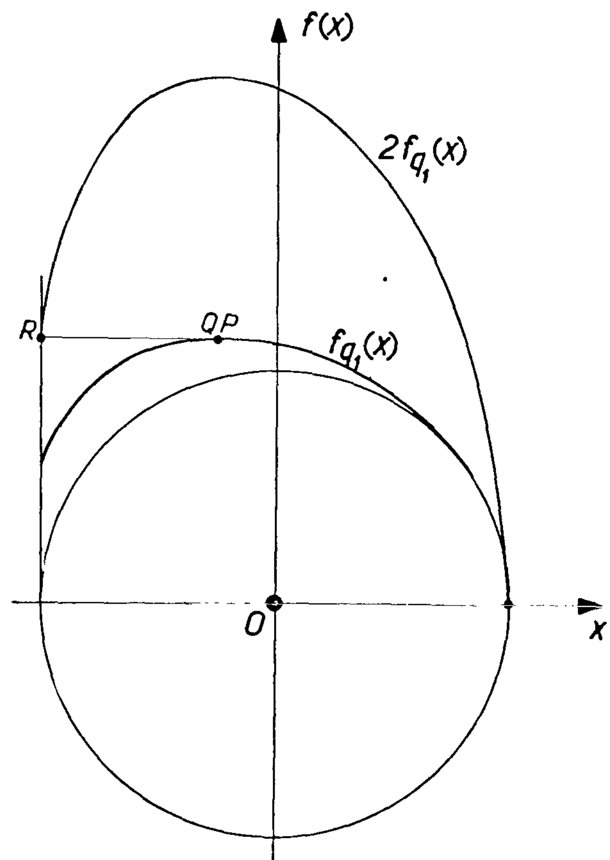

Fig. 6 - $2\pi \leqslant \Delta L \leqslant 4\pi$. $(q = q_1)$

If $0 \leq \Delta L \leq 2 \text{ Arc cos } \frac{1}{\sqrt{3}} = 109°28'$ (Figure 3a), and when q decreases, we pass successively from the bi-impulsional solution (QP) ($q = q_1 < -\sqrt{3}/2$) to solutions with two thrust arcs (QP) ($q = q_2$; $q_3 < q_2 < q_1$), then to the solution "(QP) ($q = q_3$; $-1 < q_3 < q_{1im}$).

If $2 \text{ Arc cos } \frac{1}{\sqrt{3}} < \Delta L < \overline{\Delta L}$ (Figure 3b).

Beginning with the bi-impulsional solution (RQ) ($q= q_1$) we find solutions with two thrust arcs of the type (RQ) ($q = q_2$, $q_1 < q_2 < q_3$) then of type (RP) ($q = q_4$; $q_1 < q_4 < q_3$) and finally of the type (QP) ($q = q_5$; $q_6 < q_5 < q_1$). The "continuous thrust" solution (QP) is obtained for $q = q_6$.

If $\overline{\Delta L} < \Delta L < \pi$ (Figure 3c).

Beginning with the bi-impulsional solution (RQ) ($q = q_1$) we find solutions with two thrust arcs of the type (RQ) ($q = q_2$, $q_1 < q_2 < q_3$), then of the type (RP)($q = q_4$; $q_3 < q_4 < q_5$), and finally the "continuous thrust" solution (RP) ($q = q_5$). This case does not essentially differ from the preceding case, except by the fact that we do not find a solution of the type (QP).

If $\pi < \Delta L <$ Arc cos (-1/3) (Figure 3d).

For $q \to -\infty$, we find the "Hohmann type" solution (RQ). But this is only one solution among all the possible solutions (degeneracy). /227

Then appear solutions with two thrust arcs of the type (RQ) for $q = q_1$ ($-\infty < q_1 < q_2$), then of the type (RP) for $q = q_3$ ($q_4 < q_3 < q_2$), and finally the "continuous thrust" solution (RP)for ($q = q_4$).

If 2 Arc cos (-1/3) $< \Delta L < 2\pi$ (Figure 3e).

Only solutions of the type (RQ) are presented for $-\infty < q_1 < q_2 < -1$.

2. $\boxed{2\pi \leqslant \Delta L \leqslant 4\pi}$

SOLUTIONS CORRESPONDING TO A PARTICULAR VALUE OF q and to different values of the transfer angle ΔL and of the relationship $|\lambda_1| = |\Delta a|/F_{max}$.

Let us begin with the case $q \to +\infty$. We shall obtain solutions taking part in the totality of singular solutions referring to the degenerated problem (Figure 4a). The curve $2f_{+\infty}(x)$ is an elliptical arc transformed by affinity with the arc of circle $f_{+\infty}(x)$.

The angles L_o and L_1 are connected by:

$$\sin L_o = 2 \sin L_1. \tag{1}$$

The solutions of the type (RQ) (Figure 4b), [R(QP)] (Figure 4c) and (RP) (Figure 4d) are arranged in a series from the "continuous thrust" solution on one revolution RQ = 0 ($\Delta L = 2\pi$) up to the "continuous thrust" solution on two revolutions (RP) ($\Delta L = 4\pi$).

For $q > q_1$ (Figure 5), the solutions are also of the type (RQ) and (RP) with two "continuous thrust" solutions.

For $q = q_1$ (Figure 6), there is a single solution [R(QP)] corresponding to "continuous thrust".

3. FOR $\boxed{2N\pi \leqslant \Delta L \leqslant 2(N+1)\pi}$,

q is $\gtrless 0$ depending on whether N is $\begin{cases}\text{odd} \\ \text{even}\end{cases}$ and there are N + 2 thrust arcs.

For $q = \pm\infty$, the generalizing condition (1) is:

$$\sin L_o = (N + 1) \sin L_1.$$

APPENDIX 13

OPTIMAL IMPULSIONAL TRANSFERS BETWEEN ~~CLOSE~~, NEAR-CIRCULAR ORBITS, CO-PLANAR OR NOT (see Chapter II,5)

1. TYPE I (Bi-impulsional). /229

Let us take the axis of reference $\vec{\vec{Ox}}$ according to $\vec{\vec{p}}_e$. The impulses $I'\Delta C$ and $I''\Delta C(I' + I'' = 1)$ applied to points $M'(L')$ and $M''(L'' = -L')$ cause the variations:

$$\left\{\begin{aligned}
&\Delta \xi = Z' \sin L' \Delta C \\
&\Delta \eta = -Z' \cos L' (I' - I'') \Delta C \\
&\Delta a = 2Y' \Delta C \\
&\Delta \alpha = (X' \sin L' + 2Y' \cos L') \Delta C \\
&\Delta \beta = (-X' \cos L' + 2Y' \sin L')(I' - I'') \Delta C
\end{aligned}\right. \tag{1}$$

We shall suppose that $\sin L' > 0$ for the impulse $I'\Delta C$.

From these equations it is easy to deduce the equations (II,5. 5-9).

Therefore the optimal thrust law is expressed as a function of the transfer parameters ($\vec{\vec{\Delta j}}$, $\Delta e_{//}$, $\Delta e_{\perp}$, Δa) and of δ and ΔC, which are unknown for the moment. In order to determine these two unknowns, we have two equations available:

The first expresses the fact that X', Y', Z' are guiding cosines:

$$X'^2 + Y'^2 + Z'^2 = 1. \tag{2}$$

The second connects X', Y', Z' and the position L'.

$$tg\, L' = \frac{1 - Y'^2}{2X'Y'} = \frac{X'^2 + Z'^2}{2X'Y'}. \tag{3}$$

This last equation is obtained geometrically in the following way (Figure 1).

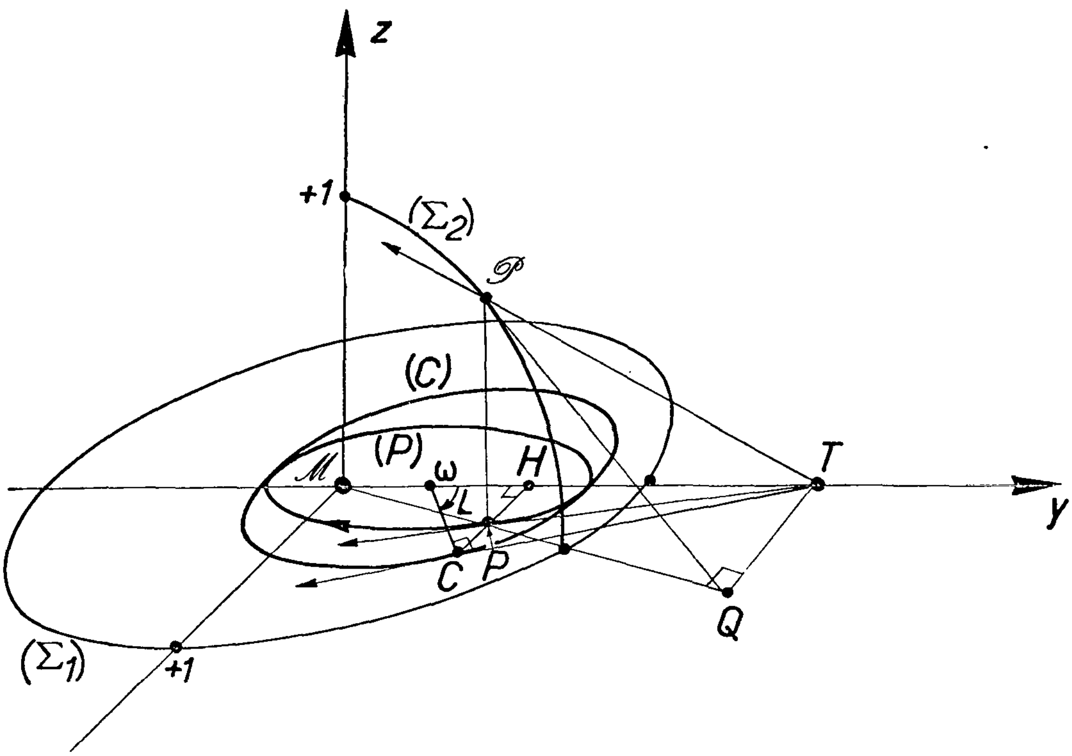

Fig. 1.

Let $P(X,Y,Z)$ be one of the contact points of the ellipse (P) of the sphere (Σ). The tangent $\overrightarrow{TP}$ in P to (P) is the intersection of the plane (π) of (P), defined by the point P and the axis $\overrightarrow{MY}$, and of the plane tangent in P to (Σ), defined by P and its course QT in the plane $Z = 0$. Therefore the straight line TP is tangent to the projection (P) of (P) on the plane $Z = 0$. The ellipse (P) is unique and well determined, by the datum of its major axis $\overrightarrow{MY}$, its eccentricity $e = \sqrt{3}/2$ and its tangent $\overrightarrow{TP}$ in P. By an orthogonal affinity of axis $\overrightarrow{MY}$ and of equation 2, (P) is actually transformed into a circle (C), centered on $\overrightarrow{MY}$ and tangent in C to $\overrightarrow{TC}$. This circle is unique and centered in /230
ω.

$\mathcal{P}P$ is the polar of Q in reference to the major circle (Σ_2) of (Σ), therefore QT is the polar of P in reference to the major circle (Σ_1) of (Σ) , and finally PH is the polar of τ in reference to (Σ_1), and therefore: $MH \cdot MT = 1$

Whence:

$$\cot g\left(\frac{\pi}{2} - L\right) = tg\ L = \frac{HT}{HC} = \frac{MT - MH}{HC} = \frac{\frac{1}{Y} - Y}{2X} = \frac{1 - Y^2}{2XY}.$$

Introducing the values of X', Y' and L' into (3), we derive

$$tg\,\delta = \frac{\Delta e_{//}^2 - \frac{3}{4}\Delta a^2 - \Delta C^2}{\Delta e_{//}\,\Delta e_{\perp}} \tag{4}$$

as a function of the consumption ΔC, the latter being itself furnished by the biquadratic equation:

$$\boxed{\left(-\Delta e_{//}^2 + \frac{3}{4}\Delta a^2 + \Delta C^2\right)\left(\Delta e_{\perp}^2 + \Delta j^2 + \frac{\Delta a^2}{4} - \Delta C^2\right) + \Delta e_{\perp}^2\left(\Delta e_{//}^2 - \Delta a^2\right) = 0} \tag{5}$$

obtained by introducing the values of X', Y' and Z' into equation (2). /231

tg δ can also be derived directly from the second degree equation (II,5-4).

The inequalities (II,5-6) are satisfied if:

$$\Delta e_{//}^2 \leqslant \Delta a^2. \tag{6}$$

The solutions of type I correspond to points of the plane $\Delta e_{//}\Delta e_{\perp}$ in the interior of the belt $|\Delta e_{//}| < |\Delta a|$ (Figure 2).

The inequalities (II,5-5) are satisfied if:

$$tg^2\delta \geqslant \frac{\Delta a^2}{\Delta e_{//}^2} - 1 = tg\delta'\, tg\delta'' \geqslant 0 \tag{7}$$

where tgδ' and tgδ'' are the roots of (II,5-4).

Therefore it is necessary that the square of the root (II,5-4) adopted be superior or equal to the products of the roots. This is possible for only a single root of this equation, the one of which the modulus is maximal.

As the numerator of the expression (4) of tgδ is negative (for

$$Y^2 < 1 \Longrightarrow \Delta C^2 \geqslant \frac{\Delta a^2}{4}),$$

this condition shows that the consumption ΔC is given by *the greatest* of the roots of the biquadratic equation (5).

Finally, the useful angle condition

$$|z| \geqslant \sqrt{3}\,|x| \ : \tag{8}$$

is written in the limiting case

$$|\Delta e_{//}|\,|\,tg\,\delta| = \frac{\sqrt{3}\,(\Delta e_{//}^2 - \Delta a^2)}{|\Delta j| - \sqrt{3}\,|\Delta e_\perp|}$$

or, by introduction into (4)

$$\left(\frac{|\Delta j|}{\sqrt{3}} - |\Delta e_\perp|\right)\left(\sqrt{3}\,|\Delta j| + |\Delta e_\perp|\right) = \Delta e_{//}^2 - \Delta a^2$$

or even:

$$\Delta e_{//}^2 + \left(|\Delta e_\perp| + \frac{|\Delta j|}{\sqrt{3}}\right)^2 = \frac{4}{3}\,\Delta j^2 + \Delta a^2. \tag{9}$$

The frontier in the plane $\Delta e_{//}\Delta e_\perp$ is formed by the circle arcs (σ+) and (σ-). It is easily verified that the origin $O(\Delta e_{//} = \Delta e_\perp = 0)$ satisfies the inequality (8) and therefore corresponds to solutions of type I.

2. TYPE II (nodal bi-impulsional).

Let us take the reference axis $\vec{O}\vec{x}$ according to the nodal line. The impulses $I'\Delta C$ and $I''\Delta C (I' + I'' = 1)$ applied to points $M'(L' = 0)$ and $M''(L'' = \pi)$ causes the variations:

$$\begin{cases} \Delta \xi = 0 \\ \Delta \eta = -|\vec{\Delta j}| = -Z'\Delta C \\ \Delta a = 2Y'(I' - I'')\Delta C \\ \Delta \alpha = \Delta e_{//} = 2Y'\Delta C \\ \Delta \beta = \Delta e_\perp = -X'\Delta C \end{cases} \tag{10}$$

Whence the relations (II,5. 11-14). /233

The inequalities (II,5 - 11) are satisfied if

$$\Delta e_{//}^2 \geqslant \Delta a^2. \tag{11}$$

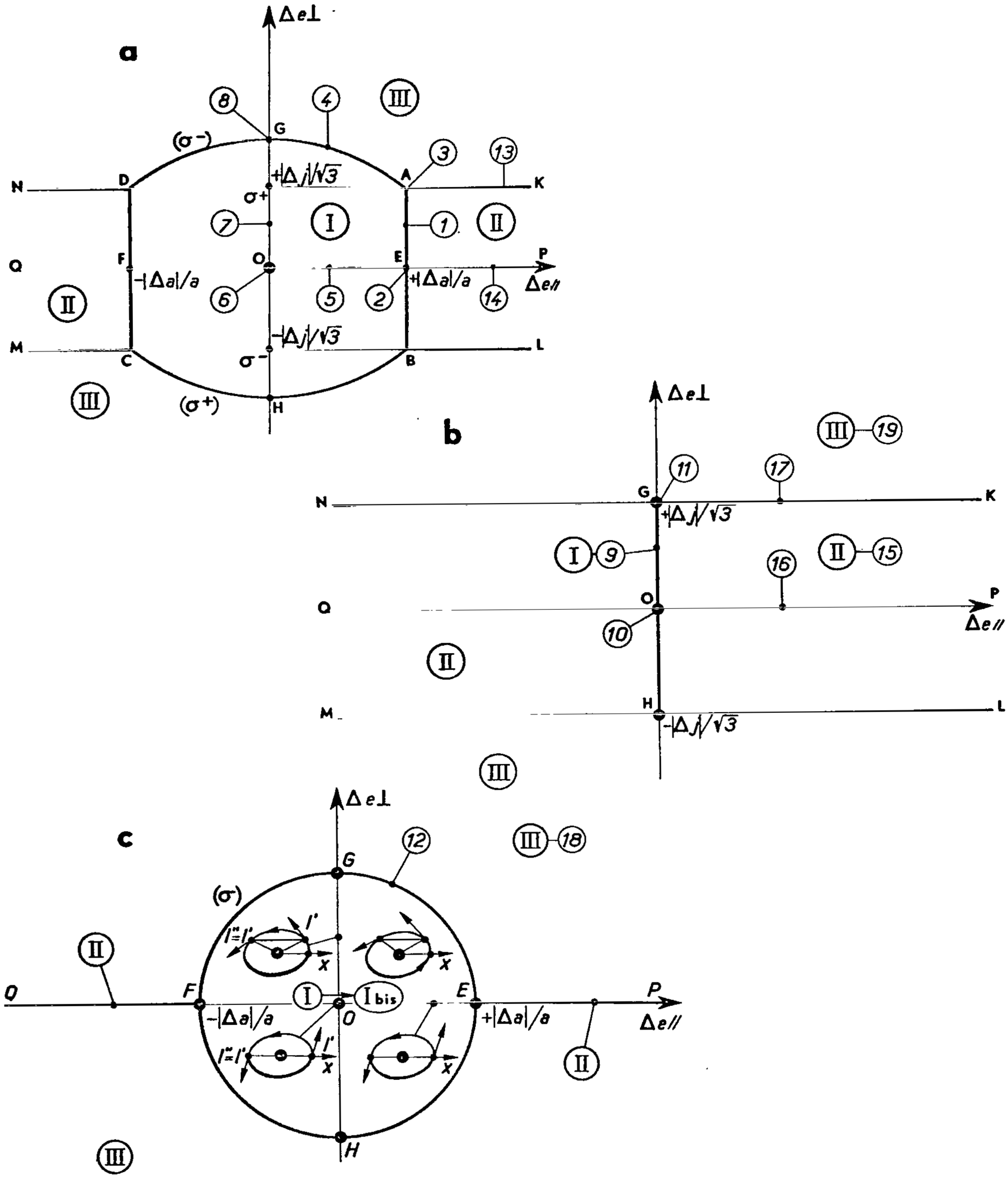

Fig. 2. Determination of the Type of Transfer.
a: General case; b: Δa = 0; c: Δj → 0.

The solutions of type II correspond to points of the plane $\Delta e_{//}\Delta e_{\perp}$ located necessarily *outside* of the belt $|\Delta e_{//}| < |\Delta a|$ (Figure 2).

The useful angle condition (8) is written here:

$$|\Delta e_{\perp}| \leqslant \frac{|\Delta j|}{\sqrt{3}}. \tag{12}$$

Therefore the point $\Delta e_{//}\Delta e_{\perp}$ is inside the belt defined by (12).

Finally writing that X'Y'Z' are guiding cosines, $(X'^2 + Y'^2 + Z'^2 = 1)$, we obtain the consumption

$$\boxed{\frac{\Delta e_{//}^2}{4} + \Delta e_{\perp}^2 + \Delta j^2 = \Delta C^2} \tag{13}$$

3. TYPE III (Singular, three-dimensional).

In order that G may be inside the volume (*V*) limited by the cylinders (C_I) and C_{II}), it is necessary for W (projection of G on the plane $z_1 = 0$) (Figure II,5 - 12) to be inside circle (C), or:

$$\Delta\xi^2 + \Delta\eta^2 \mp \frac{\sqrt{3}}{2}\Delta\xi\Delta C = \Delta j^2 \pm \frac{\sqrt{3}}{2}\overrightarrow{\Delta j_1}.\overrightarrow{\Delta C} \leqslant 0 \tag{14}$$

and that U (projection of G on plane $x_1 = 0$) be situated in the concave part of the parabola (P):

$$\frac{\Delta a^2}{4} \leqslant \Delta C^2 \mp \frac{2}{\sqrt{3}}\Delta\xi\Delta C = \Delta C^2 \pm \frac{2}{\sqrt{3}}\overrightarrow{\Delta j_1}.\overrightarrow{\Delta C} \tag{15}$$

in the region:

$$y_1 = \pm\frac{2}{\sqrt{3}}\Delta\xi = \mp\overrightarrow{\Delta j_1}.\frac{\overrightarrow{\Delta C}}{|\overrightarrow{\Delta C}|} \geqslant 0. \tag{16}$$

Let us multiply the vectorial equation (II,5 - 19) by $\overrightarrow{\Delta j}_1$:

$$\overrightarrow{\Delta j_1}.\overrightarrow{\Delta e} \mp \sqrt{3}\,\overrightarrow{\Delta j_1}^2 = \left|\overrightarrow{\Delta j_1}\right|\Delta e_{\perp} \mp \sqrt{3}\,\overrightarrow{\Delta j_1}^2 = 2\overrightarrow{\Delta C}.\overrightarrow{\Delta j_1}. \tag{17}$$

Introducing this value of $\overrightarrow{\Delta C}.\ \overrightarrow{\Delta j}_1$ into inequality (14) it becomes:

$$\mp\sqrt{3}\,\Delta e_\perp \geqslant |\overrightarrow{\Delta j_1}| \geqslant 0 \qquad (18)$$

therefore, at first,

$$\boxed{\pm = -\,\mathrm{sign}\ \Delta e_\perp} \qquad (19)$$

which permits a choice to be made between the orientations (π^+) and (π^-), then:

$$|\Delta e_\perp| \geqslant \frac{|\overrightarrow{\Delta j}|}{\sqrt{3}}. \qquad (20)$$

In the plane $\Delta e_{//}\Delta e_\perp$, the solutions of type III correspond to points /234
necessarily outside of the belt

$$|\Delta e_\perp| \leqslant \frac{|\overrightarrow{\Delta j}|}{\sqrt{3}} \quad \text{(Figure 2).}$$

The consumption ΔC is obtained by raising the vectorial equation (II,5-19) to the square

$$\boxed{\Delta C^2 = \frac{\Delta e_{//}^2}{4} + \frac{(|\Delta e_\perp| + \sqrt{3}\,|\Delta j|)^2}{4}} \qquad (21)$$

Finally introducing the calculated values of $\overrightarrow{\Delta j}_1$, $\overrightarrow{\Delta C}$ and of ΔC^2 into (15), this inequality becomes:

$$\left(|\Delta e_\perp| + \sqrt{3}\,|\overrightarrow{\Delta j}|\right)\left(|\Delta e_\perp| - \frac{|\overrightarrow{\Delta j}|}{\sqrt{3}}\right) \geqslant \Delta a^2 - \Delta e_\perp^2. \qquad (22)$$

Again we find frontier (9).

4. MONO-IMPULSIONAL SOLUTIONS.

So that the preceding study, dedicated to multi-impulsional cases, also covers the totality of impulsional solutions, it is enough to demonstrate that any mono-impulsional solution at all can always be considered as a limiting case of a bi-impulsional solution when one of the two impulses becomes 0.

Let $\overrightarrow{MP}$ be any mono-impulsional solution (Figure 3).

The corresponding ellipse (P) is inside the sphere (Σ) and tangent to this /235
sphere in P. The ellipse (P), a projection of (P) on plane $Z = 0$, is inside

the ellipse (P_{III}), a projection on z = 0 of the circles (C^+) and (C^-) (a condition imposed by the "useful dihedral").

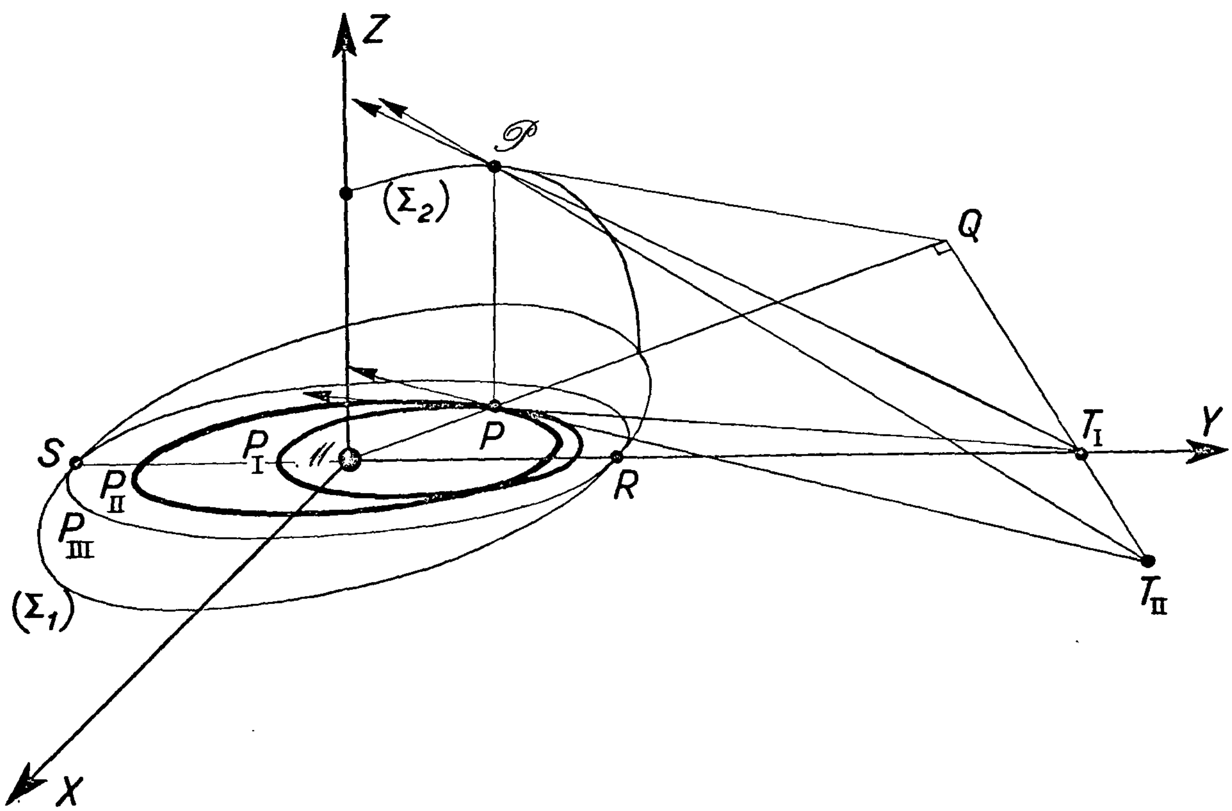

Fig. 3.

In general there exist two "efficiency" ellipses (P_I) and (P_{II}), interiorly bi-tangent to (Σ'), one of the contact points being P, referring to solutions respectively of type I and type II.

The projection of (P_I) on the plane Z = 0 is the ellipse $(P_I)(\frac{b}{a} = \frac{1}{2})$ of major axis $\overrightarrow{MY}$, tangent in P to the straight line PT_I. The cylinder (σ_I) is therefore well determined. Moreover, (P_I) is contained in the plane (π_I) passing through $\overrightarrow{MY}$ and P. It is tangent to the straight line PT_I in P.

The projection of (P_{II}) on plane z = 0 is the ellipse $(P_{II})(\frac{b}{a} = \frac{1}{2})$ of major axis $\overrightarrow{MY}$ and of minor axis $\overrightarrow{MX}$, passing through point P. Therefore the cylinder (σ_{II}) is well determined. (P_{II}) is contained in the plane (π_{II})

passing through P and MT_{II}. It is tangent to the straight line PT_{II} in P.

Thus, every mono-impulsional solution can be considered as a limiting case either of a bi-impulsional solution of type I, or of a bi-impulsional solution of type II, when one of the two impulses becomes 0. (Circle arc $\overset{\frown}{PN}$ of Figure II,5 - 11b).

When point P of Figure 3 is on P_{III}, the two efficiency curves (P_I) and (P_{II}) coincide with one of the circles (C^+) or (C^-). The impulsional solution [one impulse in one of the planes (π^+) or (π^-)] can then be considered as the limiting case of solutions of type I, II or III ("triple" point N of Figure II,5 - 11b).

In particular point R (or S) corresponds to the mono-impulsional tangent solutions (circle arc $\overset{\frown}{F_2M_2}$ of Figure II,5 - 11c).

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